RECENT DEVELOPMENTS IN THE THEORY OF TENSOR EIGENVECTORS

HIROTACHI ABO

ABSTRACT. Eigenvectors of tensors, an extension of eigenvectors of matrices, were introduced by L.-H. Lim [5] and L. Qi [8] independently in 2005 and have been studied in numerical multi-linear algebra (see, for example, [5, 8, 6, 4]). Recently, the concept of an eigenvector of a tensor drew attention to the algebraic geometry community because algebraic geometry is proven to provide useful techniques for the tensor eigenproblem. This is a preliminary report of two on-going projects related to schemes, whose associated reduced schemes are sets of eigenvectors of tensors, called *eigenschemes* that started during the semester-long thematic program "AGATES: Algebraic Geometry with Applications to TEnsors and Secants." The first is concerned with the the discrete classification of eigenschemes of tensors, and the second is to extend the concepts of the sets of the right- and left-eigenvectors of matrices to tensors as well as to explore the compatibility of such concepts.

1. INTRODUCTION

A non-zero vector is called an *eigenvector* of a matrix if the vector and its image under the matrix are linearly dependent. This means that one can write the conditions for a non-zero vector to be an eigenvector of a matrix in terms of the maximal minors of the matrix obtained by concatenating these two vectors horizontally. We call the scheme defined by such minors the *eigenscheme* of the matrix.

The geometry of the eigenscheme of a matrix gives an alternative interpretation of the diagonalizability of the matrix. In fact, the eigenscheme of a diagonalizable matrix and its associated reduced scheme (which we call the *eigenvariety* of the matrix) share the same eigeninformation; while if the matrix is not diagonalizable, then the eigenscheme and the eigenvariety are different and their difference makes the Jordan structure of the matrix visible. For instance, the decomposition of the eigenscheme into primary components contains the numeric data of the Jordan matrix of the matrix, such as the number of Jordan blocks and the size of each Jordan block. Therefore, the correspondence between matrices and eigenschemes enables us to construct a dictionary between linear algebra and algebraic geometry.

The concept of eigenvectors of matrices was extended to tensors by L.-H. Lim [5] and L. Qi [8] independently. As in the case of matrices, the conditions for a non-zero vector to be an eigenvector of a tensor can be described in terms of polynomials. Thus, one, analogously, can define the eigenscheme of a tensor as a (not necessarily integral) scheme given by such polynomials. My primary interest is in

Key words and phrases. Tensors, eigenvectors of tensors, schemes, tangent bundles.

seeking a tensor analog of the linear algebra-algebraic geometry dictionary as well as to build a bridge between multi-linear algebra and algebraic geometry through the study of algebro-geometric aspects of tensor eigenvectors.

The purpose of this manuscript is to report progress of two on-going projects on eigenschemes of tensors, both of which came out of during (or just prior to) the semester-long thematic program "AGATES: Algebraic Geometry with Applications to TEnsors and Secants." (1) The first project is to give a discrete classification of eigenschemes of tensors, and (2) the second project is about the compatibility of eigenschemes of a tensor (a generalization of the compatibility between the set of right eigenvectors and the set of left eigenvectors of a matrix).

(1) An ultimate goal is to classify eigenschemes up to isomorphism. This project is concerned with a discrete part of this classification problem.

The classification is in terms of partitions of integers. If the eigenscheme Z_A of a tensor $A \in (\mathbb{C}^{n+1})^{\otimes d}$ is of dimension 0, then the length of Z_A is known to be $\delta := \sum_{i=0}^{n}(d-1)^i$ (see [3, 7] for the proof). Suppose that $Z_A = Z_0 \cup Z_1 \cup \cdots \cup Z_k$ is the decomposition into irreducible components. If λ_i denotes the length of each irreducible component Z_i of Z_A, then $\boldsymbol{\lambda} := (\lambda_0, \lambda_1, \dots, \lambda_k)$ forms a partition of δ, which we call the *partition* of Z_A. (For example, if Z_A is non-singular of dimension 0, then Z_A has the partition whose parts are all one.) The primary goal of this project is to explore the question as to whether for each partition $\boldsymbol{\lambda}$ of δ, there exists a tensor $A \in (\mathbb{C}^{n+1})^{\otimes d}$ whose eigenscheme Z_A has partition $\boldsymbol{\lambda}$.

(2) An $(n+1)\times(n+1)$ matrix A and its transpose A^T share the same eigenvalues, but they do not necessarily share the same eigenvectors. There exists, however, the "compatibility" between the set of eigenvectors of A and the set of eigenvectors of A^T (or equivalently the set of left eigenvectors of A).

Assume that A has $n+1$ distinct eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_n$, and let $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be eigenvectors corresponding to $\lambda_0, \lambda_1, \dots, \lambda_n$ respectively. If B is the $(n+1) \times (n+1)$ matrix whose ith column is $\boldsymbol{v}_i$, then $B^{-1}AB = D$, where D is the diagonal matrix with diagonal entries $\lambda_0, \lambda_1, \dots, \lambda_n$. The columns of $(B^{-1})^T$ are eigenvectors of A^T. This implies that if $S := \{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $T := \{\boldsymbol{w}_0, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ are two sets of $n+1$ linearly independent vectors, then S and T are the sets of right- and left-eigenvectors of A if and only if, after a suitable relabeling, $\boldsymbol{w}_i^T \boldsymbol{v}_j = 0$ whenever $i \neq j$.

The compatibility between the sets of the right- and the left-eigenvectors of the matrix can be expressed in terms of polynomial equations. The variety defined by these polynomials is called the *eigencompatibility variety* for matrices.

The concepts of the sets of the right- and the left-eigenvectors of a matrix can be extended to a tensor by flattening it in various ways, and the compatibility of such concepts can also be translated into polynomial equations. Thus, one, analogously, can define the eigencompatibility variety for tensors. The primary goal of this project is two-fold; the first is to determine the dimensions of the eigencompatibility varieties for tensors, and the second is to find their equations.

This manuscript is organized as follows. In Section 2, we review the basic definitions and facts of tensor eigenvectors and tensor eigenschemes. In Section 3, we discuss an alternative definition of the eigenscheme of a tensor, which represents the eigenscheme of the tensor as the zero locus of a global section of a certain vector bundle. In Section 4, we take a first step towards classifying the eigenschemes in terms of partitions of integers, starting with the ternary cubic tensors (i.e., $n = 2$ and $d = 3$). In Section 5, the formula for the dimension of the eigencompatibility variety for binary tensors will be given. The equations for such an eigencompatibility variety will also be discussed.

2. EIGENSCHEMES OF TENSORS

Throughout this manuscript, we denote by V an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $S := \{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ and by V^* the dual of V. Let $\mathbb{P}V$ be the projective space of lines of V passing through the origin. If $\boldsymbol{v} \in V$ is non-zero, then we denote by $[\boldsymbol{v}]$ the equivalence class of $\boldsymbol{v}$. Let $V^{\otimes d}$ be the tensor product of d copies of V.

For each $\boldsymbol{i} := (i_1, i_2, \ldots, i_d) \in \Lambda := (\mathbb{Z}_n)^d$, we write $\boldsymbol{e_i}$ for $\boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_d} \in V^{\otimes d}$ and $\boldsymbol{x_i}$ for the monomial $\prod_{\alpha=1}^{d} x_{i_\alpha}$. Let

$$A := \sum_{\boldsymbol{i} \in \Lambda} a_{\boldsymbol{i}} \, \boldsymbol{e_i}$$

and let $A\,\boldsymbol{x}^{d-1}$ be the element of $V \otimes \mathrm{Sym}_{d-1}(V^*) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}_{d-1}(V), V)$ defined by

$$A\,\boldsymbol{x}^{d-1} := \sum_{\alpha=0}^{n} \boldsymbol{e}_\alpha \otimes [A\,\boldsymbol{x}^{d-1}]_\alpha,$$

where

$$[A\,\boldsymbol{x}^{d-1}]_\alpha := \sum_{0 \le i_2 \le \cdots \le i_d \le n} a_{\alpha i_2 \ldots i_d} x_{i_2} \cdots x_{i_d}.$$

Definition 2.1. A non-zero vector $\boldsymbol{v} \in V$ is called an *E-eigenvector* (or just an *eigenvector*) of A if $A\,\boldsymbol{v}^{d-1} = \lambda\,\boldsymbol{v}$ for some $\lambda \in \mathbb{C}$.

Remark 2.2. Let $\boldsymbol{x} = \sum_{i=0}^{n} x_i \boldsymbol{e}_i$ and let $[\boldsymbol{x}]_S$ be the coordinate vector of $\boldsymbol{x}$ with respect to S. If $d = 2$, then A can be considered as a linear transformation from V to V^*. If $[A]_S$ denotes the matrix representation of A with respect to S and the dual basis associated with S, then $A\,\boldsymbol{x}^1 = [A]_S\,[\boldsymbol{x}]_S$. Hence the concept of eigenvector of a tensor can be thought of as a natural extension of the concept of an eigenvector of a matrix.

Let $A \in V^{\otimes d}$ and let $\boldsymbol{v} \in V$ be an eigenvector of A. By definition, the vectors $A\,\boldsymbol{v}^{d-1}$ and $\boldsymbol{v}$ are linearly dependent. This means that $\boldsymbol{v} \in V$ is an eigenvector of A if and only if the coefficients of $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$, $0 \le i < j \le n$, in $A\,\boldsymbol{x}^{d-1} \wedge \boldsymbol{x}$, vanish at $\boldsymbol{v}$. It also follows immediately that a non-zero $\boldsymbol{v} \in V$ is an eigenvector of A if and only if a non-zero scalar multiple of $\boldsymbol{v}$ is an eigenvector of A. We, therefore, can unambiguously define an "eigenpoint" of A in $\mathbb{P}V$ as $[\boldsymbol{v}]$.

Definition 2.3. An element $[\boldsymbol{v}]$ of $\mathbb{P}V$ is called an *eigenpoint* of A if it lies in the algebraic subset in $\mathbb{P}V$ defined by the homogeneous ideal I_A generated by the coefficients of $A\,\boldsymbol{x}^{d-1} \wedge \boldsymbol{x}$ of $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$, $0 \le i < j \le n$, . We call the set of eigenpoints of A the *eigenvariety* of A and the closed subscheme of $\mathbb{P}V$ defined by I_A the *eigenscheme* of A. We write V_A and Z_A for the eigenvariety and eigenscheme of A respectively.

Remark 2.4. If $A \in V^{\otimes d}$, then, identifying with its coordinate vector with respect to S, $A\,\boldsymbol{x}^{d-1}$ can be considered as the vector of homogeneous polynomials of degree $d-1$. Therefore, it can also be viewed as a polynomial rational map from $\mathbb{P}V$ to $\mathbb{P}V$ itself. Thus, the eigenpoints of A is, by definition, either a fixed point of $A\,\boldsymbol{x}^{d-1}$ or a point where $A\,\boldsymbol{x}^{d-1}$ is undefined.

If the eigenscheme Z_A of A is non-singular of dimension 0, then the length of Z_A is known.

Theorem 2.5 ([3, 7])**.** *If the eigenscheme Z_A of $A \in V^{\otimes d}$ is zero dimensional, then the length of Z_A is*

$$\frac{(d-1)^{n+1}-1}{d-2} = \sum_{i=0}^{n}(d-1)^i. \tag{2.1}$$

Remark 2.6. The above formula was first given in [6] when d is even. In the same paper, it was suggested the same formula holds for arbitrary d. In [3], D. Cartwright and B. Sturmfels solved this conjecture using techniques from toric geometry. In [7], L. Oeding and G. Ottaviani also obtained the same formula using the vector bundle approach, which we will illustrate in the next section.

3. Eigenschemes via vector bundles

The main purpose of this subsection is two-fold: to show that the eigenscheme of a tensor can be viewed as the zero scheme of a global section of a certain vector bundle, and to describe how to use the language of vector bundles to study eigenschemes of tensors.

In addition to the notation introduced in Section 2, we also use the following symbols:

(1) $\mathscr{O} := \mathscr{O}_{\mathbb{P}V}$ the sheaf of rings on $\mathbb{P}V$;
(2) $\mathscr{O}(1)$ the hyperplane bundle on $\mathbb{P}V$;
(3) $\mathscr{O}(-1)$ the dual bundle of $\mathscr{O}(1)$;
(4) $\mathscr{O}(d)$ the tensor product of $|d|$ copies of $\mathscr{O}(1)$ (resp. $\mathscr{O}(-1)$) if $d \geq 0$ (resp. $d < 0$).

If F is a coherent sheaf on $\mathbb{P}V$, then $F(d)$ denotes $F \otimes \mathscr{O}(d)$, and $H^iF := H^i(\mathbb{P}V, F)$ denotes the ith cohomology group of F.

Let $K^\bullet$ be the Koszul complex, i.e., the complex $\{K^\alpha, \wedge^\alpha \boldsymbol{x}\}_{0\leq\alpha\leq n}$ with

$$K^\alpha := \bigwedge^{\alpha} V \otimes \mathscr{O}\,(d+\alpha-2)\,,$$

and $\wedge^\alpha \boldsymbol{x} : K^\alpha \to K^{\alpha+1}$ is determined by

$$\wedge^\alpha \boldsymbol{x}\,(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_\alpha}) := \sum_{j=0}^{n} x_j\,\boldsymbol{e}_j \wedge \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_\alpha}.$$

Note that $\wedge^0 \boldsymbol{x} = \boldsymbol{x}$ and $\wedge^1 \boldsymbol{x} = \wedge \boldsymbol{x}$.

The tangent bundle $T(d-2) := T_{\mathbb{P}V}(d-2)$ twisted by $d-2$ is obtained as the cokernel of $\boldsymbol{x}$, i.e.,

$$0 \longrightarrow \mathcal{O}(d-2) \xrightarrow{\boldsymbol{x}} V \otimes \mathcal{O}(d-1) \xrightarrow{\wedge \boldsymbol{x}} T(d-2) \longrightarrow 0. \tag{3.1}$$

Note that

$$\begin{aligned} H^0\mathcal{O}(d-2) &= \mathrm{Sym}_{d-2}(V^*), \\ H^0(V \otimes \mathcal{O}(d-1)) &= V \otimes \mathrm{Sym}_{d-1}(V^*), \\ H^1\mathcal{O}(d-2) &= 0. \end{aligned}$$

Thus, taking the cohomology yields the following exact sequence:

$$0 \longrightarrow \mathrm{Sym}_{d-2}(V^*) \xrightarrow{\boldsymbol{x}} V \otimes \mathrm{Sym}_{d-1}(V^*) \xrightarrow{\wedge \boldsymbol{x}} H^0T(d-2) \longrightarrow 0. \tag{3.2}$$

Notice that the image of $A\,\boldsymbol{x}^{d-1}$ under $\wedge\boldsymbol{x}$ equals the exterior power of $A\,\boldsymbol{x}^{d-1}$ and $\boldsymbol{x}$. Therefore, there exists a natural isomorphism

$$H^0T(d-2) = \{A\,\boldsymbol{x}^{d-1} \wedge \boldsymbol{x} \,|\, A \in V^{\otimes d}\}.$$

from which it follows that a non-zero vector $\boldsymbol{v} \in V$ is an eigenvector of A if and only if $[\boldsymbol{v}] \in \mathbb{P}V$ is a point of the zero scheme of the corresponding global section of $T(d-2)$. In other words, a subscheme of $\mathbb{P}V$ is the eigenscheme of a tensor of $V^{\otimes d}$ if and only if it is the zero scheme of some global section of $T(d-2)$.

Remark 3.1.
(1) From the short exact sequence (3.2) it follows that every element of $\mathbb{P}V$ is an eigenpoint of $A \in V^{\otimes d}$ if and only if $A\,\boldsymbol{x}^{d-1}$ is a scalar multiple of $\boldsymbol{x}$ by a non-zero element of $\mathrm{Sym}_{d-2}(V^*)$.
(2) Since $T(d-2)$ is globally generated, the zero scheme of a generic global section of $T(d-2)$ is non-singular of codimension n. In particular, if $A \in V^{\otimes d}$ is generic, then the zero scheme of the corresponding global section of $T(d-2)$ is reduced and has codimension n. Furthermore, its length coincides with the top Chern class of $T(d-2)$, which is $\sum_{i=0}^{n}(d-1)^i$ (see Theorem 2.5.)
(3) The tensors in $V^{\otimes d}$ whose eigenscheme is non-singular of codimension n form an open subset of the projective space of $V^{\otimes d}$. The complement of such an open subset (denoted $\Delta_{n,d}$) is called the *eigendiscriminant*, which is proven to be an irreducible hypersurface of degree $(n+1)n(d-1)^n$ (see [2] for more details).

4. Classification of eigenschemes in terms of partitions

Suppose that the eigenscheme Z_A of $A \in V^{\otimes d}$ has dimension 0. Let δ denote the length of Z_A, so that $\delta = \sum_{i=0}^{n}(d-1)^i$, and let Z_A have a decomposition

$$Z_A = Z_0 \cup Z_1 \cup \cdots \cup Z_k$$

into irreducible components. We write λ_i for the length of each primary component Z_i. By reordering $Z_0, Z_1, \ldots, Z_k$ if necessary, we can associate to Z_A the partition $\boldsymbol{\lambda} := (\lambda_0, \lambda_1, \ldots, \lambda_k)$ of δ, and we call $\boldsymbol{\lambda}$ the *partition* of Z_A.

Example 4.1. (1) If $A \in V^{\otimes d}$ is generic (i.e., $A \notin \varDelta_{n,d}$), then Z_A is non-singular of codimension 0 (see Remark 3.1), which means that each primary component of Z_A is a reduced closed point, and hence Z_A has partition $(\underbrace{1, 1, \cdots, 1}_{\delta})$.

(2) Recall that the eigendiscriminant $\varDelta_{n,d}$ parameterizes tensors whose eigenscheme is "singular" (i.e., it is singular of codimension n or has a positive dimensional component). If $[A] \in \varDelta_{n,d}$ is generic, then the partition of Z_A is $(2, \underbrace{1, 1, \ldots, 1}_{\delta-1})$.

(3) Let

$$A := \sum_{i=0}^{n-1} \boldsymbol{e}_{i+1} \otimes \underbrace{\boldsymbol{e}_i \otimes \cdots \otimes \boldsymbol{e}_i}_{d-1} \in V^{\otimes d}.$$

Since $A\,\boldsymbol{x}^{d-1} = \boldsymbol{e}_1 \otimes x_0^{d-1} + \boldsymbol{e}_2 \otimes x_1^{d-1} + \cdots + \boldsymbol{e}_n \otimes x_{n-1}^{d-1}$, the ideal I_{Z_A} of Z_A is generated by the following set:

$$\left\{x_0^d\right\} \cup \left\{x_0 x_i^{d-1} \;\middle|\; 1 \le i \le n-1\right\} \cup \left\{x_i^{d-1} x_{j+1} - x_{i+1} x_j^{d-1} \;\middle|\; 0 \le i < j \le n-1\right\}.$$

Let P be the point defined by $x_0 = x_1 = \cdots = x_{n-1} = 0$. If I_P denotes the ideal of P, then one can prove that I_{Z_A} contains the $(2d-1)(n-1)$th power of I_P, which implies that I_{Z_A} is I_P-primary. Therefore, Z_A is a zero-dimensional closed subscheme of length δ supported at P, and hence (δ) is the partition of Z_A.

Example 4.1 leads us to the question as to whether, for each of the remaining partitions λ of δ, there exists a tensor $A \in V^{\otimes d}$ whose eigenscheme Z_A has partition $\boldsymbol{\lambda}$.

Remark 4.2. There are two cases, where the answer is known. The first is the case, where $n = 1$, and the other is the case, where $d = 2$.

If $n = 1$, then every zero-dimensional closed subscheme of length $\delta = d$ is the eigenscheme of some order-d binary tensor (see [2, Remark 4.2] for more details). Therefore, for any partition $\boldsymbol{\lambda}$ of d, there exists an order-d binary tensor whose eigenscheme has partition $\boldsymbol{\lambda}$. If $d = 2$ and if $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_k)$ is a partition of $\delta = n+1$, then the eigenscheme of the matrix whose Jordan canonical form consists of k Jordan blocks of sizes $\lambda_0, \lambda_1, \ldots, \lambda_k$ has partition $\boldsymbol{\lambda}$ (see [1] for more details).

The following theorem, which is a part of the ongoing project with F. Galuppi, F. Gesmundo, P. Santarsiero, T. Seynnaeve, affirmatively answers the question for the first non-trivial case.

Theorem 4.3. *If $n = 2$ and $d = 3$, then, for every partition $\boldsymbol{\lambda}$ of $\delta = 7$, there exists a tensor $A \in V^{\otimes d}$ whose eigenscheme has partition $\boldsymbol{\lambda}$.*

Remark 4.4. The proof of Theorem 4.3 is by explicit construction. More precisely, for each partition $\boldsymbol{\lambda}$ of 7, we constructed an order-3 ternary tensor whose eigenscheme has partition $\boldsymbol{\lambda}$.

5. COMPATIBILITY OF EIGENCONFIGURATIONS

Let V be an $(n+1)$-dimensional vector space over $\mathbb{C}$ and let $S := \{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be a basis. As in Section 2, for each $\boldsymbol{i} := (i_1, i_2, \ldots, i_d) \in \Lambda := (\mathbb{Z}_n)^d$, we write $\boldsymbol{e_i}$ for $\boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_d} \in V^{\otimes d}$ and $\boldsymbol{x_i}$ for the monomial $\prod_{\alpha=1}^{d} x_{i_\alpha}$.

To each tensor $A = \sum_{\boldsymbol{i}\in\Lambda} a_{\boldsymbol{i}}\, \boldsymbol{e_i} \in V^{\otimes d}$, we assign the following element of $V \otimes \mathrm{Sym}_{d-1} V^*$:

$$A^{(\alpha)}\, \boldsymbol{x}^{d-1} = \sum_{\boldsymbol{i}\in\Lambda} \boldsymbol{e}_{i_\alpha} \otimes a_{\boldsymbol{i}}\ (\boldsymbol{x_i}/x_{i_\alpha}) = \sum_{\boldsymbol{i}\in\Lambda} \boldsymbol{e}_{i_\alpha} \otimes a_{\boldsymbol{i}}\, x_{\boldsymbol{i}_1} \cdots x_{\boldsymbol{i}_{\alpha-1}} x_{\boldsymbol{i}_{\alpha+1}} \cdots x_{\boldsymbol{i}_d}. \tag{5.1}$$

Note that $V \otimes \mathrm{Sym}_{d-1} V^* = \mathrm{Hom}\,(\mathrm{Sym}_{d-1} V, V)$. For each $j \in \{0, \ldots, n-1\}$, let $[A^{(\alpha)}\, \boldsymbol{x}^{d-1}]_j$ be the jth coordinate of $A^{(\alpha)}\, \boldsymbol{x}^{d-1}$ when considered as an element of $\mathrm{Hom}\,(\mathrm{Sym}_{d-1} V, V)$. In concrete term,

$$\left[A^{(\alpha)}\, \boldsymbol{x}^{d-1}\right]_j = \sum_{\boldsymbol{i}\in\Lambda_{\alpha,j}} a_{\boldsymbol{i}}\, x_{i_1} \cdots x_{i_{\alpha-1}} x_{i_{\alpha+1}} \cdots x_{i_d},$$

where $\Lambda_{\alpha,j}$ is the set of d-tuples of elements of $\mathbb{Z}_n$ whose αth component is j.

Definition 5.1. For each $\alpha \in \{1, 2, \ldots, d\}$, let $I_{A,\alpha}$ be the ideal generated by the coefficients of $\boldsymbol{e_i}$, $\boldsymbol{i} \in \Lambda$, in $A^{(\alpha)}\, \boldsymbol{x}^{d-1} \wedge \boldsymbol{x}$. The scheme defined by $I_{A,\alpha}$ is called the αth eigenscheme of A and denoted by $Z_{A,\alpha}$. We call a closed point of $Z_{A,\alpha}$ an αth *eigenpoint* of A.

Remark 5.2.
1. If $A \in V^{\otimes d}$ such that $Z_{A,\alpha}$, then, by Theorem 2.5, then the length of $Z_{A,\alpha}$ is $D(n,d) := \sum_{i=0}^{n} (d-1)^i$.
2. As is mentioned in Remark 3.1, the eigenscheme $Z_{A,\alpha}$ of each $A \in V^{\otimes d}$ can be interpreted as the zero scheme of a global section $s_{A,\alpha}$ of $T(d-2)$ for each $\alpha \in \{1, 2, \ldots, d\}$.

Definition 5.3. For each $\alpha \in \{1, 2, \ldots, d\}$, define a map $\varphi_\alpha : \mathbb{P}V^{\otimes d} \to \mathbb{P}H^0 T(d-2)$ by sending $[A]$ to $[s_{A,\alpha}]$, and let $\varphi := (\varphi_1, \varphi_2, \ldots, \varphi_d) : \mathbb{P}V^{\otimes d} \to (\mathbb{P}H^0 T(d-2))^{\oplus d}$. We call the image of $\mathbb{P}V^{\otimes d}$ under φ the *eigencompatibility variety* and denote it by $\mathrm{EC}_{n,d}$.

Example 5.4. We consider the case where $n = 1$ and $d = 2$. If

$$A := \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix},$$

then $A^{(1)}\, \boldsymbol{x}^1 = A\, \boldsymbol{x}$ and $A^{(2)}\, \boldsymbol{x}^1 = A^T\, \boldsymbol{x}$. In particular,

$$\begin{aligned} A^{(1)}\, \boldsymbol{x}^1 \wedge \boldsymbol{x} &= -a_{10}x_0^2 + (a_{00} - a_{11})x_0 x_1 + a_{01} x_1^2, \\ A^{(2)}\, \boldsymbol{x}^1 \wedge \boldsymbol{x} &= -a_{01}x_0^2 + (a_{00} - a_{11})x_0 x_1 + a_{10} x_1^2. \end{aligned}$$

Thus, the map $\varphi : \mathbb{P}V^{\otimes d} \to (\mathbb{P}H^0 T)^{\oplus 2} = (\mathbb{P}H^0 \mathscr{O}(2))^{\oplus 2}$ is given by

$$\varphi([A]) = ([-a_{10} : a_{00} - a_{11} : a_{01}], [-a_{01} : a_{00} - a_{11} : a_{10}]).$$

If $([u_0 : u_1 : u_2], [v_0 : v_1 : v_2])$ denotes the bi-homogeneous coordinates of $(\mathbb{P}H^0 \mathscr{O}(2))^{\oplus 2}$, then $\mathrm{EC}_{1,2}$ is defined by the 2×2 minors of the following matrix:

$$\begin{bmatrix} u_0 & u_1 & u_2 \\ -v_2 & v_1 & -v_0 \end{bmatrix}.$$

This implies that $\mathrm{EC}_{1,2}$ is the double Veronese embedding of $\mathbb{P}H^0\mathscr{O}(2)$. In particular, $\dim \mathrm{EC}_{1,2} = 2$.

The following theorem, which is a part of the ongoing collaboration with J. Buczyński and A. Woo, solved the problem of finding the dimension of the eigencompatibility variety for the binary case, which was suggested in [2].

Theorem 5.5.

$$\dim \mathrm{EC}_{1,d} = \begin{cases} d^2 - d + 1 & \textit{if } d \textit{ is odd,} \\ d^2 - d & \textit{if } d \textit{ is even.} \end{cases}$$

Remark 5.6. For each $\alpha \in \{1, 2, \ldots, d\}$, let $[u_0^{(\alpha)} : u_1^{(\alpha)} : \cdots : u_n^{(\alpha)}]$ be the homogeneous coordinates of the αth factor of $(\mathbb{P}H^0\mathscr{O}(d))^{\oplus d}$ and let $k := \lfloor d/2 \rfloor$. Consider the following linear forms:

$$L_0^{(\alpha)} := \sum_{t=0}^{k} (-1)^t u_{2t}^{(\alpha)}$$

and

$$L_1^{(\alpha)} := \begin{cases} \sum_{t=0}^{k-1} (-1)^t u_{2t+1}^{(\alpha)} & \text{if } d \text{ is even,} \\ \sum_{t=0}^{k} (-1)^t u_{2t+1}^{(\alpha)} & \text{if } d \text{ is odd.} \end{cases}$$

The eigencompatibility variety $\mathrm{EC}_{1,d}$ is contained in the subvariety M_d of $(\mathbb{P}H^0\mathscr{O}(d))^{\oplus d}$ defined by the 2×2 minors of the following matrix:

$$\begin{bmatrix} L_0^{(0)} & L_0^{(1)} & \cdots & L_0^{(d-1)} \\ L_1^{(0)} & L_1^{(1)} & \cdots & L_1^{(d-1)} \end{bmatrix}.$$

Note that the subvariety M_d is geometrically interpreted as follows: If $P_{\lambda\mu}^{(\alpha)}$ denotes the hyperplane of the αth factor of $(\mathbb{P}H^0\mathscr{O}(2))^{\oplus d}$ defined by $\lambda L_0^{(\alpha)} + \mu L_1^{(\alpha)}$ for each $[\lambda : \mu] \in \mathbb{P}^1$, then

$$M_d = \bigcup_{[\lambda:\mu]\in\mathbb{P}^1} (P_{\lambda\mu}^{(1)} \times P_{\lambda\mu}^{(2)} \times \cdots \times P_{\lambda\mu}^{(d)})$$

whose singular locus is defined by $L_0^{(\alpha)} = L_1^{(\alpha)} = 0$. In particular, $\dim M_d = d(d-1)+1$. Therefore, $\mathrm{EC}_{1,d} = M_d$ if d is odd, while $\mathrm{EC}_{1,d} \subsetneq M_d$ if d is even.

Acknowledgement

This work is partially supported by the Thematic Research Programme "Tensors: geometry, complexity and quantum entanglement", University of Warsaw, Excellence Initiative – Research University and the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021-2023."

REFERENCES

[1] H. Abo, D. Eklund, T. Kahle, and C. Peterson, *Eigenschemes and the Jordan canonical form*, Linear Algebra Appl. **496** (2016), 121–151.

[2] H. Abo, A. Seigal, and B. Sturmfels, *Eigenconfigurations of tensors*, Algebraic and geometric methods in discrete mathematics, 1–25, Contemp. Math., **685**, Amer. Math. Soc., Providence, RI, 2017.

[3] D. Cartwright and B. Sturmfels, *The number of eigenvalues of a tensor*, Linear Algebra Appl. **438** (2013), no. 2, 942–952.

[4] T. Kolda and J. Mayo, *Shifted power method for computing tensor eigenpairs*, SIAM J. Matrix Anal. Appl. **32** (2011), no. 4, 1095–1124.

[5] L.-H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, in CAM-SAP'05: Proceeding of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 2005, pp. 129–132.

[6] G. Ni, L. Qi, F. Wang, and Y. Wang, *The degree of the E-characteristic polynomial of an even order tensor*, J. Math. Anal. Appl. **329** (2007) 1218–1229.

[7] L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput. **54** (2013), 9–35.

[8] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005) 1302–1324.

DEPARTMENT OF MATHEMATICS AND STATISTICAL SCIENCE, UNIVERSITY OF IDAHO, MOSCOW, ID 83844-1103, USA

Email address: abo@uidaho.edu