Border rank of powers of ternary quadratic forms

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Definition (Waring rank of a polynomial $h \in \mathcal{R}_d$)

$$\operatorname{rk}(h) = \min \left\{ r \in \mathbb{N} \;\middle|\; h = \sum_{j=1}^{r} (a_{1,j}x_1 + \cdots + a_{n,j}x_n)^d \,:\, a_{i,j} \in \mathbb{C} \right\}$$

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Definition (Border rank of a polynomial $h \in \mathcal{R}_d$)

$$\operatorname{brk}(h) = \min \left\{ r \in \mathbb{N} \;\middle|\; h = \lim_{t \to 0} \sum_{j=1}^{r} \left(a_{1,j}(t)x_1 + \cdots + a_{n,j}(t)x_n\right)^d : a_{i,j}(t) \in \mathbb{C} \right\}$$

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Classical decompositions

$$\left[q_3^2\right]_7 = \frac{2}{3}\sum_j^3 x_j^4 + \frac{1}{12}\sum^4 (x_1 \pm x_2 \pm x_3)^4$$ (E. Lucas, 1877)

$$\left[q_4^2\right]_{12} = \frac{2}{3}\sum_j^4 x_j^4 + \frac{1}{24}\sum^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^4$$ (J. Liouville, 1859)

$$\left[q_n^2\right]_{n^2} = \frac{1}{6}\sum_{j_1<j_2}^{\binom{n}{2}} \left(x_{j_1} + x_{j_2}\right)^4 + \frac{1}{6}\sum_{j_1<j_2}^{\binom{n}{2}} \left(x_{j_1} - x_{j_2}\right)^4 + \frac{4-n}{3}\sum_j^n x_j^4$$

(B. Reznick, 1992)

Theorem (B. Reznick)

$$\operatorname{rk}\left(q_2^s\right) = s + 1.$$

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Decompositions of q_2^s

$$q_2^s = \sum_{j=1}^{s+1} \left(r(s)\cos(\tau_j)x_1 + r(s)\sin(\tau_j)x_2\right)^{2s}, \qquad \tau_j = \frac{(j-1)\pi}{s+1}$$

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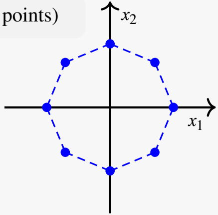
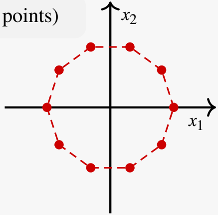
Examples

Decomposition of q_2^3 (4 points)



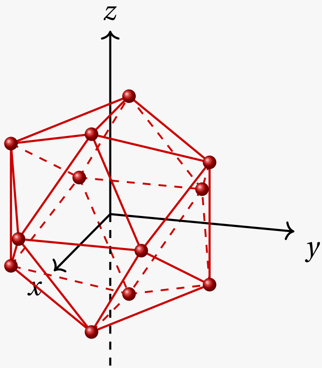
Decomposition of q_2^4 (5 points)



Three variables

Decomposition of q_3^2 (6 points)

$$[q_3^2]_6 = \frac{1}{6}\sum_j^6 (x_j \pm \varphi x_{j-1})^4, \quad \varphi = \frac{1+\sqrt{5}}{2}$$



Apolarity action

- Apolarity action of $\mathcal{D}_k$ on $\mathcal{R}_j$

$$\circ : \mathcal{D}_k \times \mathcal{R}_j \longrightarrow \mathcal{R}_{j-k}$$

$$\left(\mathbf{y}^{\alpha}, \mathbf{x}^{\beta}\right) \longmapsto \frac{\partial}{\partial \mathbf{x}^{\alpha}}\left(\mathbf{x}^{\beta}\right)$$

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Definition (Apolar ideal of a homogeneous polynomial $h \in \mathcal{R}_d$)

$$h^{\perp} = \{\, g \in S(V^*) \mid g \circ h = 0 \,\}.$$

Catalecticant map of $h \in S^d V$

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Proposition (J. J. Sylvester, 1851)

For every $f \in \mathcal{R}_d$ and $0 \leq k \leq d$

$$\mathrm{rk}\, f \geq \mathrm{brk}\, f \geq \mathrm{rk}\left(\mathrm{Cat}_f^k\right).$$

Definition (Laplace operator)

Differential operator $\Delta\colon \mathcal{D}_d \to \mathcal{D}_{d-2}$

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Space of the d-harmonic polynomials

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Theorem

$$(q_n^s)^\perp = \left(\mathcal{H}_n^{s+1}\right)$$

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Lemma (Apolarity lemma)

Let $f \in \mathcal{R}_d$ and $Z \subset \mathbb{P}^n$ a 0-dimensional scheme. Let $\nu_d : \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(S^d\mathbb{C}^n)$ be the d-Veronese map. The following conditions are equivalent:

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- $f \in \langle \nu_d(Z) \rangle$;
- $I(Z) \subseteq f^{\perp}$.

Lie algebra of $\mathrm{SL}_2\mathbb{C}$

$$\mathfrak{sl}_2\mathbb{C} = \{ A \in \mathrm{Mat}_2(\mathbb{C}) \mid \operatorname{tr} A = 0 \}, \quad \dim(\mathfrak{sl}_2\mathbb{C}) = 3.$$

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Basis of $\mathfrak{sl}_2\mathbb{C}$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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- $S^n(\mathbb{C}^2)$ essentially unique irreducible representation;

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- $\{x^{n-k}y^k\}_{k=0,\dots,n}$ set of *weights*;
- $V_k = \langle x^{n-k}y^k \rangle$ for every $k = 0, \dots, n$

$$E : V_k \longrightarrow V_{k+1}, \quad H : V_k \longrightarrow V_k, \quad F : V_k \longrightarrow V_{k-1}$$

Proposition (R. Goodman and N. R. Wallach)

The space $\mathcal{H}_n^d$ *is an irreducible* $\mathrm{SO}_n(\mathbb{C})$*-module.*

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Change of variables

$$u = \frac{y_1 + \mathrm{i}y_2}{2}, \quad v = \frac{y_1 - \mathrm{i}y_2}{2}, \quad z = y_3.$$

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Basis of $\mathfrak{so}_3\mathbb{C}$ (with respect to $\{u, v, z\}$)

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}.$$

Laplace operator

$$\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} = \frac{\partial^2}{\partial u\, \partial v} + \frac{\partial^2}{\partial z^2}.$$

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Notation (divided powers)

$$u^{[k_1]} v^{[k_2]} z^{[k_3]} = \frac{1}{k_1!k_2!k_3!} u^{k_1} v^{k_2} z^{k_3}, \qquad k_1, k_2, k_3 \in \mathbb{N}.$$

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Basis $\mathcal{B}_d = \{h_{d,k}\}_{-d \le k \le d}$ of $\mathcal{H}_3^d$

$$h_{d,k} = \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j u^{[k+j]} z^{[d-k-2j]} v^{[j]}, \quad h_{d,-k} = \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j u^{[j]} z^{[d-k-2j]} v^{[k+j]}$$

Harmonic generators of the basis $\mathcal{B}_d = \{h_{d,k}\}_{-d \le k \le d}$

$$\underbrace{\left\langle \frac{1}{2}u^2 \right\rangle}_{h_{2,2}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{4}uz \right\rangle}_{h_{2,1}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{12}\left(z^2 - 2uv\right) \right\rangle}_{h_{2,0}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{4}vz \right\rangle}_{h_{2,-1}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{2}v^2 \right\rangle}_{h_{2,-2}}$$

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$$\vdots$$

$$\underbrace{\left\langle \frac{1}{d!}u^d \right\rangle}_{h_{d,d}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{2d(d-1)!}u^{d-1}z \right\rangle}_{h_{d,d-1}} \underset{E}{\overset{F}{\rightleftarrows}} \cdots \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{2d(d-1)!}v^{d-1}z \right\rangle}_{h_{d,-(d-1)}} \underset{E}{\overset{F}{\rightleftarrows}} \underbrace{\left\langle \frac{1}{d!}v^d \right\rangle}_{h_{d,-d}}$$

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Candidate ideal

$$I_{s+1} = (h_{s+1,s+1}, \dots, h_{s+1,0}) \subset \left(q_3^s\right)^{\perp} \quad \sqrt{I_{s+1}} = (u, z) \quad \deg(I_{s+1}) = \binom{s+2}{2}$$

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Smoothable rank of $f \in \mathcal{R}_d$

$$\operatorname{smrk} f = \min \{ r \in \mathbb{N} \mid \exists \text{ 0-dim sm. scheme } Z\colon \deg Z = r, f \in \langle \nu_d(Z) \rangle \}$$

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Use of apolarity lemma

$$\text{Every scheme } Z \subseteq \mathbb{P}^2\mathbb{C} \text{ is smoothable} \Longrightarrow \operatorname{smrk}\left(q_3^s\right) \leq \binom{s+2}{2}$$

Border rank

$$\binom{s+2}{2} \leq \operatorname{brk}(q_3^s) \leq \operatorname{smrk}(q_3^s) \leq \binom{s+2}{2}$$

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Conclusion

Let $f \in \mathbb{C}[x_1, x_2, x_3]_2$. Then:

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- $\operatorname{rk} f = 3 \Rightarrow \operatorname{brk}(f^s) = \binom{s+2}{2}$

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