1 FUNDAMENTALS OF Q.M.

Postulate I: The state of an isolated system is represented by a $\underbrace{\text{state vector}}_{\text{norm } 1}$ $|\psi\rangle$ in a $\underbrace{\text{Hilbert space } \mathcal{H}}_{\text{complex, separable}}$.

examples:

qubit (2-level system / spin ½) $\mathcal{H} = \mathbb{C}^2$

Spinless particle (non-relativistic) $\mathcal{H} := L_2(\mathbb{R}^3)$

Dirac notation:

e_0, e_1 vs $|0\rangle, |1\rangle$

$e_0^*, e_1^* \in \mathcal{H}^*$ vs $\langle 0|, \langle 1|$

advantage:

$$\underset{\langle e_i, e_j\rangle}{\underset{\shortparallel}{e_i^* e_j}} \quad \text{vs} \quad \langle i | j\rangle$$

note: orth. proj. on subspace generates by $|v\rangle$ is

$$P = |v\rangle\langle v|$$

$$P|\psi\rangle = |v\rangle\langle v|\psi\rangle = \langle v|\psi\rangle \cdot |v\rangle$$

$$= \langle v, \psi\rangle \cdot |v\rangle$$

OBSERVATION: normalization $\|\psi\| = 1$ will be justified

Soon as a way to guarantee that certain measurement outcomes have total probability of 1.

At the same time, it will be clear that $e^{i\theta}|\psi\rangle$ gives the same results as $|\psi\rangle$. There is no physical experiment that distinguishes them.

OPTION (not followed here): Replace $\mathcal{H}$ by $P\mathcal{H}$

projective space of complex rays.

$P(\mathbb{C}^2) = \mathbb{C}P^1 = S^3$ Riemann sphere

complex projective line

Bloch sphere representation:

colatitude longitude

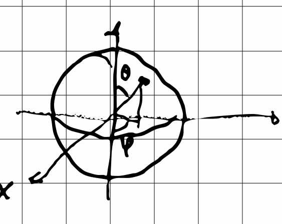
$|\psi\rangle \in \mathbb{C}^2$ $\longleftrightarrow$ spherical coordinates (θ, ϕ)

$\|\psi\| = 1$

$0 \leq \theta \leq \pi$

$0 \leq \phi < 2\pi$

$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$



$|\langle\psi_1, \psi_2\rangle|^2 = \cos^2\left(\frac{\eta}{2}\right)$ η = angle between vectors. $\in [0, \pi]$

Composite Systems: If system A is described by $\mathcal{H}_A$ and system B by $\mathcal{H}_B$,

Then the joint system AB is described by $\mathcal{H}_A \otimes \mathcal{H}_B$

$$\mathcal{H}_A = \operatorname{Span}\{|a_1\rangle, \ldots, |a_n\rangle\}$$

$$\mathcal{H}_B = \operatorname{Span}\{|b_1\rangle, \ldots, |b_m\rangle\}$$

$$\mathcal{H}_A \otimes \mathcal{H}_B = \operatorname{span}\left\{ |a_i\rangle \otimes |b_j\rangle \;\middle|\; \begin{array}{l} i=1,\ldots,n \\ j=1\ldots m \end{array} \right\}$$

notation: $|a_i\rangle \otimes |b_j\rangle = |a_i b_j\rangle$

States of the form $|\psi_A\rangle \otimes |\psi_B\rangle$ are called Product or Separable (i.e. simple tensors)

States which are not products are called entangled

examples: Bell states in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \qquad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

nomenclature: $|\Phi^+\rangle$ = EPR State (Einstein-Podolsky-Rosen)

$|\Psi^-\rangle$ = Singlet (1-dim irrep in $SU(2) \otimes SU(2)$)
invariant under $U \otimes U$

Postulate II: The closed evolution of a quantum system is described by a Unitary $\mathcal{U}$ acting on $\mathcal{H}$.

In case of a composite system, we consider the

LOCAL UNITARIES (LU): $\mathcal{U}(\mathcal{H}_A) \otimes \mathcal{U}(\mathcal{H}_B) \otimes \dots$

Postulate III: A measurement is given by a Projection-valued measure (PVM)

$$\{P_i\}_{i=1}^{n} \qquad P_i \geq 0 \quad P_i^2 = P_i \qquad \sum_i P_i = \mathbb{1}$$

$\{1, \dots, n\}$ are the possible outcomes.

$$p_i = \langle \psi | P_i \psi \rangle$$

After measuring and obtaining outcome i, the state of the system changes to

$$|\psi_i\rangle = \frac{P_i |\psi\rangle}{\| P_i |\psi\rangle \|}$$

i.e. the post-measurement state is given by an ensemble of states $\{(p_i, |\psi_i\rangle)\}_{i=1}^{n}$

Schmidt Decomposition

Teorem Let $\mathcal{H}_A$, $\mathcal{H}_B$ finite-dim. Then for every

$|\varphi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exist orthonormal sets

$\{ |u_1\rangle, \dots, |u_K\rangle \} \subseteq \mathcal{H}_A$ and

$\{ |v_1\rangle, \dots, |v_K\rangle \} \subseteq \mathcal{H}_B$ and

real, positive numbers $\lambda_1, \dots, \lambda_K$ such that.

$$|\varphi\rangle = \sum_{i=1}^{K} \lambda_i \; |u_i\rangle \otimes |v_i\rangle$$

Note: $\|\varphi\|^2 = \sum_{i=1}^{K} |\lambda_i|^2$ If $\|\varphi\| = 1$, then $\{ |\lambda_i|^2 \}_{i=1}^{K}$ are a prob. distr.

Schmidt rank: $SR(\varphi) = K$ rank of φ as a tensor in $\mathcal{H}_A \otimes \mathcal{H}_B$

If $\varphi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we have

three possible Schmidt decompositions ($A-BC$, $B-AC$, $C-AB$)

and three different Schmidt ranks r_A, r_B, r_C.

Entanglement entropy $E = - \sum_i |\lambda_i|^2 \log |\lambda_i|^2$

Shannon entropy of $\{ |\lambda_i|^2 \}_{i=1}^{K}$ $E(\varphi) \leq \log SR(\varphi)$

Mixed states

Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ some (potentially entangled) composite state.

What information about $|\Psi\rangle$ can Alice extract? What do they "see"?

If $X \in \mathcal{B}(\mathcal{H}_A)$ Hermitian, then Alice can obtain (having multiple copies of $|\Psi\rangle$) the value of

$$\omega_A(X) := \langle\Psi| X \otimes \mathbb{1}_B |\Psi\rangle$$

ω_A can be linearly extended to a state of $\mathcal{B}(\mathcal{H}_A)$: a linear, positive functional.

Therefore, there exists a density operator $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ that represents ω_A.

ρ_A positive semidefinite ($\rho_A \geq 0$), $\quad \omega_A(X) = \mathrm{Tr}(\rho_A X)$

ρ_A is called the reduced density operator of $|\Psi\rangle$

Similarly, there is a $\rho_B \in \mathcal{B}(\mathcal{H}_B)$ representing Bob's information about $|\Psi\rangle$

Prop: There exist unique linear operators

$$\mathrm{Tr}_B : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$$

$$\mathrm{Tr}_A : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$$

such that

$$\mathrm{Tr}\left((X_A \otimes \mathbb{1}_B) M_{AB} \right) = \mathrm{Tr}\left(X_A \, \mathrm{Tr}_B(M_{AB}) \right)$$

$$\mathrm{Tr}\left((\mathbb{1}_A \otimes Y_B) M_{AB} \right) = \mathrm{Tr}\left(Y_B \, \mathrm{Tr}_A(M_{AB}) \right).$$

Equivalent to

$$\mathrm{Tr}_A(X_A \otimes Y_B) = \mathrm{Tr}(X_A) \, Y_B$$

$$\mathrm{Tr}_B(X_A \otimes Y_B) = \mathrm{Tr}(Y_B) \, X_A$$

OBS $\rho_A = \mathrm{Tr}_B \underbrace{|\psi\rangle\langle\psi|}_{\text{proj on span } |\psi\rangle}$ $\qquad \rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$

(Simple exercise).

Consequence of Schmidt decomposition. $|\psi\rangle = \sum_i \lambda_i |a_i\rangle|b_i\rangle$

$$|\psi\rangle\langle\psi| = \sum_{i,j} \sqrt{\lambda_i \lambda_j} \, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$\rho_A = \sum_i \lambda_i^2 |a_i\rangle\langle a_i| \qquad \rho_B = \sum_i \lambda_i^2 |b_i\rangle\langle b_i|$$

Von Neumann entropy: $S(\rho) = -\mathrm{Tr}\, \rho \log \rho$

Entanglement entropy $=$ V.N. entropy ρ_A
$=$ V.N. " ρ_B

Density operator

$$\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \quad \in \mathcal{B}(\mathcal{H})$$

Purification $|\psi\rangle = \sum_i \sqrt{p_i}\, |\varphi_i\rangle |\eta_i\rangle \quad \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$

$|\eta_i\rangle$ o.n.b. $\mathcal{K}$.

it is always possible to purify with a copy of $\mathcal{H}$:

$$\sum_i \sqrt{p_i}\, |\varphi_i\rangle |\varphi_i\rangle$$

Purification theorem: if $|\psi\rangle$ and $|\psi'\rangle$ are purifications of ρ, there exist a unitary U such that $\mathbb{1} \otimes U |\psi\rangle = |\psi'\rangle$