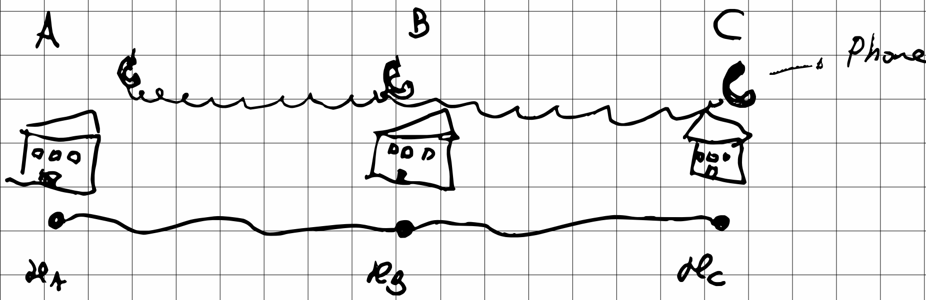
LOCC Local operations and classical communications.

Family of transformations on composite systems $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \dots$

"Distant-Lab" paradigm



Each party can only influence its own sub-system, but they are allowed to communicate the results of their measurements and other classical information.

Question: given a shared state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_B$, what other states can be obtained?

Can we classify entanglement?

2 parts entanglement

A+B share $|\psi\rangle = \sum_i \lambda_i \; |a_i\rangle |b_i\rangle \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$

Can they obtain via LOCC the state?

$$|\phi\rangle = \sum_i \mu_i \; |a_i'\rangle |b_i'\rangle$$

Teo (Nielsen)

$\psi \geq \phi$ iff. $\lambda = (\lambda_1, \dots, \lambda_s)$ majorizes $\mu = (\mu_1, \dots, \mu_s)$

Majorization: TFAE:

1) λ majorizes μ

2) $\exists$ permutation matrices P_j, $j=1 \dots n$ and prob distr $(p_j)_{j=1}^{n}$ such that

$$\mu = \sum_j p_j P_j \lambda$$

3) $\exists$ double stochastic matrix D s.t. that

$$\mu = D\lambda$$

4) If we reorder

$$\lambda^{\downarrow} = (\lambda_1^{\downarrow}, \dots, \lambda_s^{\downarrow})$$

$$\lambda_1^{\downarrow} \geq \dots \geq \lambda_s^{\downarrow}$$

Then $\sum_{i=1}^{q} \mu_i^{\downarrow} \leq \sum_{i=1}^{q} \lambda_i^{\downarrow} \quad \forall q = 1, \dots, s.$

OBS 1 Majorization is not stable under tensor products

"Catalytic conversions": $\psi \overset{LOCC}{\not\longrightarrow} \phi$ but

$\psi \otimes \tau \xrightarrow{LOCC} \phi \otimes \tau$

example $\lambda = (0.4, 0.4, 0.1, 0.1)$

$\mu = (0.5, 0.25, 0.25, 0)$

$\tau = (0.6, 0.4)$

OBS 2 Asymptotics

quant-ph/9511030

"entanglement gambling"

2 qubits bipartite state
asymptotically
entanglement entropy

Start from n copies of $\cos\theta\,|00\rangle + \sin\theta\,|11\rangle$

$$\bigotimes_{i=1}^{d} \left(\cos\theta\,|00\rangle + \sin\theta\,|11\rangle \right)$$

$$= \sum_{\vec{\alpha}} \cos\theta^{\,n-|\alpha|} \sin\theta^{\,|\alpha|} \,|\vec{\alpha}, \vec{\alpha}\rangle$$

$\vec{\alpha}$ binary string of length n

$$= \sum_{k=0}^{n} \cos\theta^{\,n-k} \sin\theta^{\,k} \underbrace{\sum_{|\vec{\alpha}|=k} |\vec{\alpha}\,\vec{\alpha}\rangle}_{\text{Subspace } V_k \text{ of dimension } \binom{n}{k}}$$

Let P_k = Proj. on subspace V_k

Measuring $\{P_k\}_{k=0}^{n}$ yields k with prob.

$$p_k = \binom{n}{k} (\cos^2\theta)^{n-k} (\sin^2\theta)^k$$

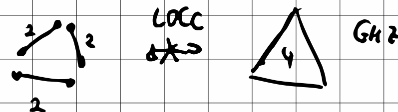
after measuring they have state with $\binom{n}{k}$ - levels EPR pairs.

expected value is $n \cdot \sin^2\theta$ asymptotically $(\sin^2\theta)^n \gg$ $\binom{n}{n/2} \sim \frac{2^n}{\sqrt{\frac{\pi}{2} n}}$

3-Parties

quant-ph/9908073

MeMu 2



$$MeMu\,2 = \frac{1}{\sqrt{2^3}}\sum_{i,j,k=0}^{1} |ij\rangle|jk\rangle|ki\rangle$$

$$|MeMu\,2\rangle\langle MeMu\,2| = \frac{1}{8}\sum_{\substack{i,j,k\\ i',j',k'}} |ij\rangle\langle i'j'| \otimes |jk\rangle\langle j'k'| \otimes |ki\rangle\langle k'i'|$$

$$Tr_A(MeMu\,2)$$

$$= \frac{1}{8}\sum_{\substack{i,j\\ k,k'}} |jk\rangle\langle jk'| \otimes |ki\rangle\langle k'i|$$

$$= \left(\frac{1}{2}\sum_{i,j} |j\rangle\langle j| \otimes |i\rangle\langle i|\right) \otimes \frac{1}{2}\sum_{k,k'} |k\rangle\langle k'| \otimes |k\rangle\langle k'|$$

$$= \frac{1}{4}\mathbb{1}_4 \otimes |EPR\rangle\langle EPR|$$

$$GHZ_4 = \frac{1}{\sqrt{4}}\sum_{i=1}^{3} |iii\rangle \qquad |GHZ_4\rangle\langle GHZ_4| = \frac{1}{4}\sum_{i,j=0}^{3} |i\rangle\langle j| \otimes |i\rangle\langle j| \otimes |i\rangle\langle j|$$

$$Tr_A\, GHZ_4 = \frac{1}{4}\sum_{i=0}^{3} |i\rangle\langle i| \otimes |i\rangle\langle i| \quad \text{"separable"}$$

__THM__: ψ, ϕ have same marginal entropies for single sites are either LU eq. or LOCC-incompatible.

__Cor 1__ LOCC eq. $\Leftrightarrow$ LU eq.

__Cor 2__ if single site marginals ~~have the~~ same entropy but not on larger systems are LOCC incompatible.

Proof $\quad \psi \xrightarrow{LOCC} \phi \quad \Rightarrow \quad \psi \xrightarrow{LU} \phi$

Step 1 of the protocol maps

$$\psi \longrightarrow \psi_i \quad \text{with prob} \quad p_i$$

$$\rho^{rest} = \sum_i p_i \, \mathrm{Tr}_A |\psi_i\rangle\langle\psi_i| \qquad \rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

$$S(\psi_A) = S(\phi_A)$$ and LOCC cannot increase entropy

$$S(\phi_{rest}) \geq \sum_i p_i \, S(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|) \geq S(\rho^{rest}) \geq S(\psi^{rest})$$

$$\Rightarrow \quad \sum_i p_i \, S(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|) = S\left(\sum_i p_i \, \mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|\right)$$

$$\Rightarrow \quad \mathrm{Tr}_A (|\psi_i\rangle\langle\psi_i|) = \mathrm{Tr}_A |\psi\rangle\langle\psi|$$

which implies that $\quad |\psi_i\rangle = U_i^A \otimes \mathbb{1}_{rest} |\psi\rangle.$

3-parties or more

quant-ph/0005115 Stocastic LOCC

"LOCC + post selection" no restriction on success probability

We consider (in place of LOCC) SLOCC (stocastic LOCC):

LOCC but only with a non-zero success probability.

This means that, when one of the parties performs a measurement $\{P_i\}_{i=1}^{n}$, we can "post select" on a given outcome, and "pretend" that it occurs with probability 1.

$|\psi\rangle \xrightarrow{\text{SLOCC}} |\varphi\rangle$ iff. $\exists$ local linear operators

$L_A, L_B, L_C, \ldots$ such that

$$L_A \otimes L_B \otimes L_C |\psi\rangle = |\varphi\rangle \qquad (*)$$

if L_A, L_B, L_C are invertible, then $|\varphi\rangle \xrightarrow{\text{SLOCC}} |\psi\rangle$, i.e. the two states are equivalent (**)

(*) is the notion of restriction, which we write as $\psi \geq \varphi$.

(**) equivalence classes are orbits of

$$GL_A \otimes GL_B \otimes GL_C.$$

quant-ph/0008031 SLOCC classification and orbits

2-parties:

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |a_i\rangle|b_i\rangle \geq \sum_{i=1}^{n} |i\rangle|i\rangle = EPR_n$$

$$= \langle n, n\rangle$$

3-parties

$\triangle\, 2 = \langle 2\rangle = GHZ_2$ v.s. $\triangle\, W = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

are SLOCC-inequivalent.

In fact let $|\psi\rangle$ a 3 qubit state with local ranks r_A, r_B, r_C

multipartite entanglement $r_A = r_B = r_C = 2$ | GHZ | W

bipartite entanglement | $r_A = 1$, $r_B = r_C = 2$ | $r_B = 1$, $r_A = r_C = 2$ | $r_C = 1$, $r_A = r_B = 2$

product states $r_A = r_B = r_C = 1$

$GHZ_2 \not\geq W$ is equivalent to Schmidt rank $W > 2$. Proof: is direct.

$W \not\geq GHZ_2$ " " " " Subrank $W > 2$

3-tangle:

$\tau(W) = 0$ and $\tau(GHZ) > 0$

Assume $|W\rangle = |a_1\rangle|b_1\rangle|c_1\rangle + |a_2\rangle|b_2\rangle|c_2\rangle$

Then $Tr_A |W\rangle\langle W|$ has range spanned by $|b_1\rangle|c_1\rangle$ and $|b_2\rangle|c_2\rangle$

Let R a 2-d subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ which is spanned by

$V_1 = |v_1^{e}\rangle \otimes |v_1^{u}\rangle$ and $V_2 = |v_2^{e}\rangle \otimes |v_2^{u}\rangle$

Then either R contains only product vectors,

or R contains no other product vectors.

case 1 excluded since local ranks are 2.

$$R = \text{Span}\{|01\rangle + |10\rangle, |00\rangle\} = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \beta \\ 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C} \right\}$$

quant-ph/0212146 quant-ph/0401023 quant-ph/0206111

3-tangle is $4\,|$ Hyperdeterminant $|$ (Cayley)

dim orbit GHZ = 7

dim orbit W = 6

dim orbit biseparable = 4

dim completely separable = 3

maximal dimension of 3 qubits.