$\boxed{1}$ Proof of the Schmidt decomposition.

Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Fix orthonormal basis

$$\mathcal{H}_A = \operatorname{Span}\{|a_1\rangle, \dots |a_n\rangle\}$$

$$\mathcal{H}_B = \operatorname{Span}\{|b_1\rangle, \dots, |b_m\rangle\}$$

And let $|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\, |a_i\rangle \otimes |b_j\rangle$

Let $C \in M_{n\times m}$ $\quad C = (c_{ij})_{i=1, j=1}^{n \quad m}$

① Let $U \in M_{n\times n}$ unitary, $V \in M_{m\times m}$, and $D = UCV^*$

Show that there exists orthonormal basis

$\{|\alpha_1\rangle, \dots, |\alpha_n\rangle\} \subseteq \mathcal{H}_A$

$\{|\beta_1\rangle, \dots, |\beta_m\rangle\} \subseteq \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}\, |\alpha_i\rangle \otimes |b_j\rangle$$

② Use the singular value decomposition of C

To construct the Schmidt decomposition of $|\psi\rangle$.

$\boxed{2}$ Prove that the following characterizations of majorization are all equivalent, so that any of them could be taken as a definition.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \quad \lambda_i \geq 0 \quad \forall i \quad \sum_i \lambda_i = 1$

$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n \quad \mu_i \geq 0 \quad \forall i \quad \sum_i \mu_i = 1$

(1) There exist a prob. distn $\{q_1, \dots, q_k\}$ (i.e. $q_j \geq 0 \quad \sum_j q_j = 1$) and permutation matrices P_j such that

$$\mu = \sum_j q_j P_j \lambda$$

(2) There exist a double stocastic matrix D such that

$$\mu = D \cdot \lambda$$

(3) Let $\lambda^{\downarrow}$ a reordering of λ such that

$$\lambda_1^{\downarrow} \geq \lambda_2^{\downarrow} \cdots \geq \lambda_n^{\downarrow}$$

Then $\displaystyle\sum_{i=1}^{q} \mu_i^{\downarrow} \leq \sum_{i=1}^{q} \lambda_i^{\downarrow} \quad \forall q = 1, \dots, n$

$\boxed{3}$ Let $\lambda = (p, 1-p)$ and $\mu = (q, 1-p)$

for some $p, q \in [0,1]$

Find conditions on p, q necessary and sufficient

for λ to majorize μ.