Fundamental invariants, latin hypercubes and rectangular Kronecker coefficients

Alimzhan Amanov
(joint work with Damir Yeliussizov)

https://arxiv.org/abs/2202.11059
AGATES Kickoff workshop, Warsaw, 2022

September 23, 2022

Content

Content

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e.
for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e. for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Denote

- $\mathbb{C}[V]$ - polynomial ring over the vector space,

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e. for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Denote

- $\mathbb{C}[V]$ - polynomial ring over the vector space,
- $\mathrm{Inv}(n_1, \ldots, n_d) := \mathbb{C}[V]^G$ - G-invariant polynomials over V,

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e. for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Denote

- $\mathbb{C}[V]$ - polynomial ring over the vector space,
- $\mathrm{Inv}(n_1, \ldots, n_d) := \mathbb{C}[V]^G$ - G-invariant polynomials over V,
- $\mathrm{Inv}(n_1, \ldots, n_d)_M$ - degree-M part of $\mathbb{C}[V]^G$.

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e.
for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Denote

- $\mathbb{C}[V]$ - polynomial ring over the vector space,
- $\mathrm{Inv}(n_1, \ldots, n_d) := \mathbb{C}[V]^G$ - G-invariant polynomials over V,
- $\mathrm{Inv}(n_1, \ldots, n_d)_M$ - degree-M part of $\mathbb{C}[V]^G$.
- In case $n_1 = \ldots = n_d = n$ we write

$$\mathrm{Inv}_d(n) := \mathrm{Inv}(n_1, \ldots, n_d)$$

.

Preliminaries: Tensor space and the group action

Consider vector space $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$.
The group $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$ acts componentwise, i.e. for $A = (A_1, \ldots, A_d) \in G$ and $v_1 \otimes \ldots \otimes v_d \in V$:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d$$

Denote

- $\mathbb{C}[V]$ - polynomial ring over the vector space,
- $\mathrm{Inv}(n_1, \ldots, n_d) := \mathbb{C}[V]^G$ - G-invariant polynomials over V,
- $\mathrm{Inv}(n_1, \ldots, n_d)_M$ - degree-M part of $\mathbb{C}[V]^G$.
- In case $n_1 = \ldots = n_d = n$ we write

  $$\mathrm{Inv}_d(n) := \mathrm{Inv}(n_1, \ldots, n_d)$$

  .
- Polynomial $P \in \mathbb{C}[V]^G$ of minimal possible degree is fundamental.

Minimal degrees

Main question: What is the minimal degree in $\mathrm{Inv}_d(n)$?

Minimal degrees

Main question: What is the minimal degree in $\mathrm{Inv}_d(n)$?

Definition

For any d and n define

$$\delta_d(n) := \min\{M : \dim \mathrm{Inv}_d(n)_M > 0\}$$

It is known:

- for any $P \in \mathrm{Inv}_d(n)_M$, then M divisible by n,
- for d-even: $\delta_d(n) = n$ (will be mentioned later)
- for d - odd: $\delta_d(n) > n$

Content

Kronecker coefficients

- For partition $\lambda \vdash m$ let $[\lambda]$ denotes Specht module, i.e. irreducible S_m representation indexed with λ.

Kronecker coefficients

- For partition $\lambda \vdash m$ let $[\lambda]$ denotes Specht module, i.e. irreducible S_m representation indexed with λ.
- Recall *Kronecker coefficients*

$$g(\lambda^{(1)}, \ldots, \lambda^{(d)}) = \text{ mult. of } [1 \times m] \text{ in } [\lambda^{(1)}] \otimes \ldots \otimes [\lambda^{(d)}].$$

Kronecker coefficients

- For partition $\lambda \vdash m$ let $[\lambda]$ denotes Specht module, i.e. irreducible S_m representation indexed with λ.
- Recall *Kronecker coefficients*
$$g(\lambda^{(1)}, \ldots, \lambda^{(d)}) = \text{ mult. of } [1 \times m] \text{ in } [\lambda^{(1)}] \otimes \ldots \otimes [\lambda^{(d)}].$$
- Alternatively, if $V_\lambda(n)$ be irreducible representation of $GL(n)$ indexed with λ then
$$g(\lambda^{(1)}, \ldots, \lambda^{(d)}) = \text{ mult. of } \bigotimes_{i=1}^{d} V_{\lambda^{(i)}}(n) \text{ in } V_{1 \times m}(n^d).$$

Kronecker coefficients

- For partition $\lambda \vdash m$ let $[\lambda]$ denotes Specht module, i.e. irreducible S_m representation indexed with λ.
- Recall *Kronecker coefficients*

$$g(\lambda^{(1)}, \dots, \lambda^{(d)}) = \text{ mult. of } [1 \times m] \text{ in } [\lambda^{(1)}] \otimes \dots \otimes [\lambda^{(d)}].$$

- Alternatively, if $V_\lambda(n)$ be irreducible representation of $GL(n)$ indexed with λ then

$$g(\lambda^{(1)}, \dots, \lambda^{(d)}) = \text{ mult. of } \bigotimes_{i=1}^{d} V_{\lambda^{(i)}}(n) \text{ in } V_{1\times m}(n^d).$$

Definition

Define sequence of rectangular Kronecker coefficients

$$g_d(n,k) := g(\underbrace{n \times k, \dots, n \times k}_{d \text{ times}}).$$

Then

$$\dim \mathrm{Inv}_d(n)_{kn} = g_d(n,k)$$

Dimensions: table d = 3,5

The table of $g_3(n,k)$ for $1 \leq k \leq 6$, $1 \leq n \leq 8$.

$k \backslash n$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0
3	1	0	1	1	1	1	0	1
4	1	1	2	5	6	13	14	18
5	1	0	1	4	21	158	1456	9854
6	1	1	3	16	216	9309	438744	17957625

The table of $g_5(n,k)$ for $1 \leq k \leq 5$, $1 \leq n \leq 6$.

$k \backslash n$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	5	11	35	52	112
3	1	1	385	44430	5942330	781763535
4	1	36	44522	381857353	5219755745322	87252488565829772
5	1	15	6008140	5220537438711	10916817688177999825	36929519748583464067841925

Rectangular Kronecker coefficients

Theorem (A.-Yeliussizov, 2022)

Let $d \geq 3$ be **odd** *and fix $k \geq 1$. Then the following properties hold:*

Rectangular Kronecker coefficients

Theorem (A.-Yeliussizov,2022)

Let $d \geq 3$ *be* **odd** *and fix* $k \geq 1$. *Then the following properties hold:*

1. *(Bounds)*
   $g_d(n,k) \leq \binom{k^d}{nk}$, *in particular* $g_d(n,k) = 0$ *for* $n \notin [0, k^{d-1}]$

Rectangular Kronecker coefficients

Theorem (A.-Yeliussizov,2022)

Let $d \geq 3$ be **odd** *and fix $k \geq 1$. Then the following properties hold:*

1. *(Bounds)*
   $g_d(n,k) \leq \binom{k^d}{nk}$, *in particular* $g_d(n,k) = 0$ *for* $n \notin [0, k^{d-1}]$
2. *(Symmetry)*
   $g_d(n,k) = g_d(k^{d-1} - n, k)$ *for* $n \in [0, k^{d-1}]$. *In particular,* $g_d(0,k) = g_d(k^{d-1}, k) = 1$.

Rectangular Kronecker coefficients

Theorem (A.-Yeliussizov,2022)

Let $d \geq 3$ be **odd** *and fix $k \geq 1$. Then the following properties hold:*

1. *(Bounds)*
   $g_d(n,k) \leq \binom{k^d}{nk}$, *in particular* $g_d(n,k) = 0$ *for* $n \notin [0, k^{d-1}]$
2. *(Symmetry)*
   $g_d(n,k) = g_d(k^{d-1} - n, k)$ *for* $n \in [0, k^{d-1}]$*. In particular,* $g_d(0,k) = g_d(k^{d-1}, k) = 1$.
3. *(Positivity)*
   $g_d(n,k) > 0, \forall n \in [0, k^{d-1}] \implies g_{d+2}(n,k) > 0, \forall n \in [0, k^{d+1}]$

Rectangular Kronecker coefficients

Theorem (A.-Yeliussizov,2022)

Let $d \geq 3$ be **odd** *and fix $k \geq 1$. Then the following properties hold:*

1. *(Bounds)*
   $g_d(n,k) \leq \binom{k^d}{nk}$, *in particular* $g_d(n,k) = 0$ *for* $n \notin [0, k^{d-1}]$
2. *(Symmetry)*
   $g_d(n,k) = g_d(k^{d-1} - n, k)$ *for* $n \in [0, k^{d-1}]$. *In particular,*
   $g_d(0,k) = g_d(k^{d-1}, k) = 1$.
3. *(Positivity)*
   $$g_d(n,k) > 0, \forall n \in [0, k^{d-1}] \implies g_{d+2}(n,k) > 0, \forall n \in [0, k^{d+1}]$$
   In particular
   $$g_3(n,k) > 0, \forall n \in [0, k^2] \implies g_d(n,k) > 0, \forall n \in [0, k^{d-1}]$$

Content

Minimal degrees: d - odd

Theorem (A.-Yeliussizov, 2022)

Let $d \geq 3$ be odd. We have:

(i) The bounds

$$n\lceil n^{1/(d-1)} \rceil \leq \delta_d(n) \leq n^2.$$

Minimal degrees: d - odd

Theorem (A.-Yeliussizov, 2022)

Let $d \geq 3$ be odd. We have:
(i) The bounds

$$n\lceil n^{1/(d-1)} \rceil \leq \delta_d(n) \leq n^2.$$

(ii) The lower bound is achieved at least in the following cases

$$\delta_d(n) = n\lceil n^{1/(d-1)} \rceil$$

for

Minimal degrees: d - odd

Theorem (A.-Yeliussizov, 2022)

Let $d \geq 3$ be odd. We have:
(i) The bounds

$$n\lceil n^{1/(d-1)} \rceil \leq \delta_d(n) \leq n^2.$$

(ii) The lower bound is achieved at least in the following cases

$$\delta_d(n) = n\lceil n^{1/(d-1)} \rceil$$

for

$$\begin{aligned} n \in & \{1, \ldots, 2^{d-1}\} \\ & \cup \{3^{d-1}, \ldots, 4^{d-1}\} \\ & \cup \{k^{d-1} - 1, k^{d-1} : k \in \mathbb{N}_{\geq 2}\} \end{aligned}$$

$[k^{d-1} - \sqrt{k}/2 + 1, k^{d-1}]$ *for even k, and also*
$\delta_d(n) \in \{3n, 4n\}$ *for* $n \in \{2^{d-1} + 1, \ldots, 3^{d-1}\}$.

Minimal degrees: d - odd

Conjecture

Let $d \geq 3$ be odd, then

$$\delta_d(n) = n\lceil n^{1/(d-1)} \rceil$$

with only exception: $d = 3$ and odd $n = k^2 - 2$ for which

$$\delta_3(k^2 - 2) = (k+1)n.$$

Content

Latin squares

Latin square L of length k - is an $k \times k$ matrix with each row and each column containing a permutation of $[k]$. Set of all Latin squares of length k denoted as $\mathcal{C}_2(k)$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}$$

Figure: Latin square of order $k = 5$

Latin squares

Latin square L of length k - is an $k \times k$ matrix with each row and each column containing a permutation of $[k]$. Set of all Latin squares of length k denoted as $\mathcal{C}_2(k)$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}$$

Figure: Latin square of order $k = 5$

For $L \in \mathcal{C}_2(k)$ the sign $\mathrm{sgn}(L)$ is a product of signs of permutations in all rows and columns.

Alon-Tarsi

The *Alon–Tarsi number* $AT(k)$ is the signed sum over Latin squares:

$$AT(k) := \sum_{L \in \mathcal{C}_2(k)} \mathrm{sgn}(L).$$

Alon-Tarsi

The *Alon–Tarsi number* $AT(k)$ is the signed sum over Latin squares:

$$AT(k) := \sum_{L \in \mathcal{C}_2(k)} \mathrm{sgn}(L).$$

It is not difficult to show that $AT(k) = 0$ for odd k.

Alon-Tarsi

The *Alon–Tarsi number* $AT(k)$ is the signed sum over Latin squares:

$$AT(k) := \sum_{L \in \mathcal{C}_2(k)} \mathrm{sgn}(L).$$

It is not difficult to show that $AT(k) = 0$ for odd k.

Conjecture (Alon–Tarsi,1992)

$$AT(k) \neq 0 \textit{ for even } k$$

Alon-Tarsi

The *Alon–Tarsi number* $AT(k)$ is the signed sum over Latin squares:

$$AT(k) := \sum_{L \in \mathcal{C}_2(k)} \mathrm{sgn}(L).$$

It is not difficult to show that $AT(k) = 0$ for odd k.

Conjecture (Alon–Tarsi,1992)

$$AT(k) \neq 0 \textit{ for even } k$$

It is known, that $AT(p \pm 1) \neq 0$ (Drisko'98, Glynn'10) where p is odd prime.

Latin hypercubes

Latin hypercube L of length k is a $k^{\times d}$ hypermatrix, such that each slice contains permutation of $[k^{d-1}]$.

For $L \in \mathcal{C}_d(k)$ define sign $\mathrm{sgn}(L)$ as product of all its permutations in slices (entries gathered in lexicographical order).

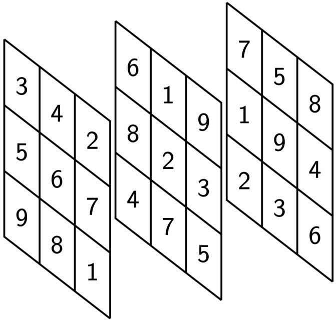

Figure: Latin (hyper)cube in $\mathcal{C}_3(3)$

Set of all Latin hypercubes of length k is denoted as $\mathcal{C}_d(k)$.

Latin hypercubes

Define d-dimensional Alon-Tarsi number

$$\mathrm{AT}_d(k) = \sum_{L \in \mathcal{C}_d(k)} \mathrm{sgn}(L)$$

Latin hypercubes

Define d-dimensional Alon-Tarsi number

$$\mathrm{AT}_d(k) = \sum_{L \in \mathcal{C}_d(k)} \mathrm{sgn}(L)$$

It is not hard to show, that when k is odd $AT_d(k) = 0$.

Latin hypercubes

Define d-dimensional Alon-Tarsi number

$$\mathrm{AT}_d(k) = \sum_{L \in \mathcal{C}_d(k)} \mathrm{sgn}(L)$$

It is not hard to show, that when k is odd $AT_d(k) = 0$.

Question (Bürgisser-Ikenmeyer, 2017, $d = 3$)

Is $\mathrm{AT}_d(k) \neq 0$ *for even* k?

Latin hypercubes

Define d-dimensional Alon-Tarsi number

$$\mathrm{AT}_d(k) = \sum_{L \in \mathcal{C}_d(k)} \mathrm{sgn}(L)$$

It is not hard to show, that when k is odd $AT_d(k) = 0$.

Question (Bürgisser-Ikenmeyer, 2017, $d = 3$)

Is $\mathrm{AT}_d(k) \neq 0$ *for even* k?

Positive answer in cases:

- $d = 3$ and $k = 2, 4$ (BI'17),
- $d \geq 3$ and $k = 2$ (AY'22).

Kronecker coefficients and Latin hypercubes

Theorem (A.-Yeliussizov, 2022)

Fix $k \geq 2$ and let $d \geq 3$ be odd. Then

$$\mathrm{AT}_d(k) \neq 0 \implies g_d(n, k) > 0 \text{ for } n \in [0, k^{d-1}].$$

Kronecker coefficients and Latin hypercubes

Theorem (A.-Yeliussizov, 2022)

Fix $k \geq 2$ and let $d \geq 3$ be odd. Then

$$\mathrm{AT}_d(k) \neq 0 \implies g_d(n, k) > 0 \text{ for } n \in [0, k^{d-1}].$$

Kronecker coefficients and Latin hypercubes

Theorem (A.-Yeliussizov, 2022)

Fix $k \geq 2$ and let $d \geq 3$ be odd. Then

$$\mathrm{AT}_d(k) \neq 0 \implies g_d(n,k) > 0 \text{ for } n \in [0, k^{d-1}].$$

For any odd d:

Corollary

$\mathrm{AT}_3(k) \neq 0$ *implies* $g_d(n,k) > 0$ *for* $n \in [0, k^{d-1}]$.

Corollary

$\mathrm{AT}_3(k) \neq 0$ *implies* $\delta_d(n) = n\lceil n^{1/(d-1)} \rceil$ *for* $n \in (\,(k-1)^{d-1}, k^{d-1}\,]$.

Content

Invariant polynomials: spanning set

Definition

Let $X \in V$ given as a hypermatrix.

Invariant polynomials: spanning set

Definition

Let $X \in V$ given as a hypermatrix. Define the polynomials $\{\Delta_T\}$ indexed by $d \times nk$ balanced table $T = (T_{i,j})$ where

$$T_i = \text{permutation of a word } 1^n \dots k^n$$

as

Invariant polynomials: spanning set

Definition

Let $X \in V$ given as a hypermatrix. Define the polynomials $\{\Delta_T\}$ indexed by $d \times nk$ balanced table $T = (T_{i,j})$ where

$$T_i = \text{permutation of a word } 1^n \dots k^n$$

as

$$\Delta_T(X) := \sum_{\sigma_1,\dots,\sigma_d:[nk]\to[n]} \mathrm{sgn}_{T_1}(\sigma_1)\dots\mathrm{sgn}_{T_d}(\sigma_d) \prod_{i=1}^{M} X_{\sigma_1(i),\dots,\sigma_d(i)}.$$

where

$$\mathrm{sgn}_{T_i}(\sigma_i) := \prod_{x=1}^{k} \mathrm{sgn}(\ \sigma_i(\{\text{positions of letter } i\})\ ) \in \{-1, 0, 1\}.$$

Invariant polynomials: spanning set

Definition

Let $X \in V$ given as a hypermatrix. Define the polynomials $\{\Delta_T\}$ indexed by $d \times nk$ balanced table $T = (T_{i,j})$ where

$$T_i = \text{permutation of a word } 1^n \dots k^n$$

as

$$\Delta_T(X) := \sum_{\sigma_1,\dots,\sigma_d:[nk]\to[n]} \mathrm{sgn}_{T_1}(\sigma_1)\dots\mathrm{sgn}_{T_d}(\sigma_d) \prod_{i=1}^{M} X_{\sigma_1(i),\dots,\sigma_d(i)}.$$

where

$$\mathrm{sgn}_{T_i}(\sigma_i) := \prod_{x=1}^{k} \mathrm{sgn}(\ \sigma_i(\{\text{positions of letter } i\})\ ) \in \{-1,0,1\}.$$

Proposition (Burgisser-Garg-Oliveira-Walter-Wigderson'17)

The polynomials $\{\Delta_T\}$ indexed by $d \times M$ balanced tables **span** *the space $\mathrm{Inv}(n_1,\dots,n_d)_M$. If n_i does not divide M for some i, this space is empty.*

Invariant polynomials: Examples

Example (Cayley's first hyperdeterminant)

A simple-looking fundamental invariant arises as follows. Let T be the $d \times n$ balanced table of all ones

$$T = \begin{pmatrix} 11\ldots 1 \\ \ldots \\ 11\ldots 1 \end{pmatrix}.$$

Then we have $\Delta_T \in \mathrm{Inv}_d(n)_n$ given as:

$$\Delta_T(X) = \sum_{\sigma_1,\ldots,\sigma_d \in S_n} \mathrm{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} X_{\sigma_1(i),\sigma_2(i),\ldots,\sigma_d(i)}.$$

This function is nontrivial when d is even, otherwise it is identically 0. This invariant was introduced by Cayley [**?**].

Invariant polynomials: Examples

Example (Cayley's second hyperdeterminant)

Let $d = 3$, $n = 2$ and consider the following balanced table:

$$T = \begin{pmatrix} 1122 \\ 1122 \\ 1212 \end{pmatrix}.$$

Then we have $\Delta_T \in \mathrm{Inv}_3(2)_4$ given as:

$$\begin{aligned} -\frac{1}{2}\Delta_T(X) = & X_{111}^2 X_{222}^2 + X_{112}^2 X_{221}^2 + X_{121}^2 X_{212}^2 + X_{211}^2 X_{122}^2 \\ & - 2(X_{111}X_{112}X_{221}X_{222} + X_{111}X_{121}X_{212}X_{222} + X_{111}X_{211}X_{122}X_{222} \\ & + X_{112}X_{121}X_{212}X_{221} + X_{112}X_{211}X_{122}X_{221} + X_{121}X_{211}X_{122}X_{212}) \\ & + 4(X_{111}X_{122}X_{212}X_{221} + X_{112}X_{121}X_{211}X_{222}). \end{aligned}$$

- Δ_T is unique fundamental invariant of $\mathrm{Inv}(2,2,2)$;
- Generates the ring $\mathrm{Inv}(2,2,2)$;
- Special case of *geometric hyperdeterminant* studied by Gelfand-Kapranov-Zelevinsky.

Invariant polynomials: Examples

Let d be odd.

Example (n^2 degree invariant, for $n = k$)

Let T be $d \times n^2$ balanced table:

$$T = \begin{pmatrix} 1^n & \dots & n^n \\ 1^n & \dots & n^n \\ \dots & \dots & \dots \\ e_n & \dots & e_n \end{pmatrix} \implies \Delta_T(I_n) = AT_2(k).$$

where $x^n = x \dots x$(n times) and $e_n = 12 \dots n$. Polynomial is candidate for $\Delta_T \in \mathrm{Inv}_d(n)_{n^2}$, but it might be zero.

Invariant polynomials: Examples

Let d be odd.

Example (k^d degree invariant, $n = k^{d-1}$)

Let T be $d \times k^d$ table of all k^d possible columns ordered lexicographically. For example, $d = 3$ and $k = 2$:

$$T = \begin{pmatrix} 11112222 \\ 11221122 \\ 12121212 \end{pmatrix}$$

Define $F_{d,k} := \Delta_T \in \mathrm{Inv}_d(k^{d-1})_{k^d}$.

It is known, that

- $F_{d,k} \neq 0$ (since $g_d(k^d, k) = 1$ and column argument)

Invariant polynomials: Examples

Let d be odd.

Example (k^d degree invariant, $n = k^{d-1}$)

Let T be $d \times k^d$ table of all k^d possible columns ordered lexicographically. For example, $d = 3$ and $k = 2$:

$$T = \begin{pmatrix} 11112222 \\ 11221122 \\ 12121212 \end{pmatrix}$$

Define $F_{d,k} := \Delta_T \in \mathrm{Inv}_d(k^{d-1})_{k^d}$.

It is known, that

- $F_{d,k} \neq 0$ (since $g_d(k^d, k) = 1$ and column argument)
- $F_{d,k}$ is fundamental invariant of a ring $\mathrm{Inv}_d(k^{d-1})$

Invariant polynomials: Examples

Let d be odd.

Example (k^d degree invariant, $n = k^{d-1}$)

Let T be $d \times k^d$ table of all k^d possible columns ordered lexicographically. For example, $d = 3$ and $k = 2$:

$$T = \begin{pmatrix} 11112222 \\ 11221122 \\ 12121212 \end{pmatrix}$$

Define $F_{d,k} := \Delta_T \in \mathrm{Inv}_d(k^{d-1})_{k^d}$.

It is known, that

- $F_{d,k} \neq 0$ (since $g_d(k^d, k) = 1$ and column argument)
- $F_{d,k}$ is fundamental invariant of a ring $\mathrm{Inv}_d(k^{d-1})$
- $F_{d,k}(I_{k^{d-1}}) = AT_d(k)$ evaluates to d-dimensional Alon-Tarsi number.

Content

Implications

Consider the following four statements:

- $A_{d,k} \; : \; F_{d,k}(I_{k^{d-1}}) \neq 0$
  (d-dim Alon-Tarsi)

- $B_{d,k} \; : \; g_d(n,k) > 0$ for all $n \leq k^{d-1}$
  (Kronecker row positivity)

- $C_{d,k} \; : \; \delta_d(n) = nk$ for all $(k-1)^{d-1} < n \leq k^{d-1}$
  (Degree lower bound is archived)

- $C'_{d,k} \; : \; \delta_d(n) \in \{nk, n(k+1)\}$ for all $(k-1)^{d-1} < n \leq k^{d-1}$
  (Degree lower bound is almost archived)

Implications

Consider the following four statements:

- $A_{d,k} \ : \ F_{d,k}(I_{k^{d-1}}) \neq 0$
  (d-dim Alon-Tarsi)
- $B_{d,k} \ : \ g_d(n,k) > 0$ for all $n \leq k^{d-1}$
  (Kronecker row positivity)
- $C_{d,k} \ : \ \delta_d(n) = nk$ for all $(k-1)^{d-1} < n \leq k^{d-1}$
  (Degree lower bound is archived)
- $C'_{d,k} \ : \ \delta_d(n) \in \{nk, n(k+1)\}$ for all $(k-1)^{d-1} < n \leq k^{d-1}$
  (Degree lower bound is almost archived)

We prove, that for every odd $d \geq 3$ the following implications hold:

$$\begin{array}{ccccc} A_{3,k} & \Longrightarrow & B_{3,k} & & \\ & & \Downarrow & & \\ A_{d,k} & \Longrightarrow & B_{d,k} & \Longrightarrow & C_{d,k} \text{ and } C'_{d,k-1} \end{array}$$

Content

Minimal degrees: d - even

Proposition

For even d we have $\delta_d(n) = n$.

There is a unique fundamental invariant $\det \in \mathrm{Inv}_d(n)_n$, which is known as *combinatorial (or Cayley's first) hyperdeterminant*:

$$\det(T) := \sum_{\sigma_1,\ldots,\sigma_d \in S_n} \mathrm{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} T_{i,\sigma_2(i),\ldots,\sigma_d(i)}$$

Analogs of Laplace expansion, Cauchy-Binnet formulas are known.

Minimal degrees: d - even

Proposition

For even d we have $\delta_d(n) = n$.

There is a unique fundamental invariant $\det \in \mathrm{Inv}_d(n)_n$, which is known as *combinatorial (or Cayley's first) hyperdeterminant*:

$$\det(T) := \sum_{\sigma_1,\dots,\sigma_d \in S_n} \mathrm{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} T_{i,\sigma_2(i),\dots,\sigma_d(i)}$$

Analogs of Laplace expansion, Cauchy-Binnet formulas are known.

Proposition (Laplace expansion)

Let d be even. For any $T \in (\mathbb{C}^n)^{\otimes d}$ we have

$$\det(T) = \sum_{i_2,\dots,i_d \in [n]} (-1)^{1+i_2+\dots+i_d} T_{1,i_2,\dots,i_d} \det(T_{\bar{1},\dots,\bar{i_d}})$$

where $\bar{i} = [n] \backslash \{i\}$ and $T_{I_1,\dots,T_d} = \{T_{i_1,\dots,i_d}\}_{i_a \in I_a}$.

Hence, if all k-minors vanish then all $(k+1)$-minors also vanish.

Minimal degrees: d - even

Problem

When $\det(T) = 0$*?*

Minimal degrees: d - even

Problem

When $\det(T) = 0$*?*

Theorem (A.-Yeliussizov'21)

For $T \in (\mathbb{C}^n)^{\otimes d}$ *if* $\det(T) \neq 0$ *then* $\operatorname{srank}(T) = n$, $\operatorname{rank}(T) \geq n$.

But it is possible that $\det(T) = 0$ and $\operatorname{srank}(T) = n$.

Thank you for your attention!