The Subrank of Random Tensors

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Joint work with Harm Derksen and Visu Makam.

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- The subrank was introduced by Strassen in 1987 to study fast matrix multiplication algorithms in CS
- and has connections to several problems in math and physics.
- Our result: We determine the subrank for "random tensors"/ "almost all tensors"/ generic tensors
- Improve on previous bounds of Strassen & Bürgisser from 1991

1. Subrank and Applications

2. Tensor Parameters and Their Value on Random Tensors

3. Subrank of Random Tensors

4. Upper bound

5. Lower bound

6. Tensor Space Decomposition

ingredient

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Two characterizations of rank of a matrix $M \in \mathbb{F}^{n \times n}$

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Decomposition into simple matrices

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$$M = \sum_{i=1}^{r} u_i \otimes v_i$$

Equiv:

$$M = A \, I_r \, B$$

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Gaussian elimination

$$A M B = I_r$$

maximize

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$$T = U \otimes V \otimes W \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

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$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \qquad R(T)$$

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Subrank

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$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T \qquad Q(T)$$

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Applications

- Matrix multiplication
- Circuit complexity [Raz]

Subrank

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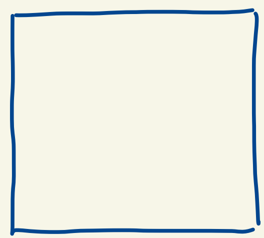
$Q(T)$

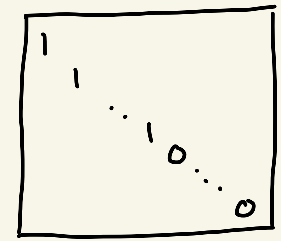
- Matrix Multiplication
- Additive Combinatorics

Matrix rank

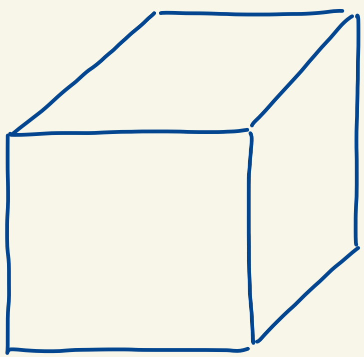
$M =$ 

linear combinations
of rows and columns

$\rightsquigarrow$

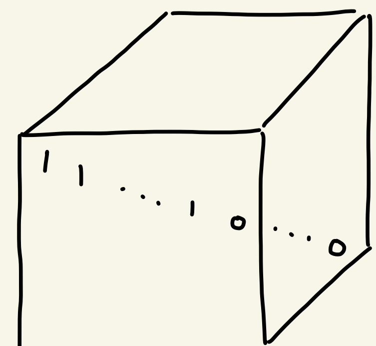


Subrank

$T =$ 

linear combinations
of slices in all
three directions

$\rightsquigarrow$



Applications of Subrank

- Complexity Theory

$T \in \mathbb{F}^{n \times n \times n} \quad \leftrightsquigarrow \quad$ bilinear map $T : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$

$Q(T) \quad \leftrightsquigarrow \quad$ number of independent scalar multiplications that can be reduced to T

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- Quantum Information

$T \in \mathbb{C}^{n \times n \times n}$ $\rightsquigarrow$ Tripartite quantum state

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- Combinatorics

$H \subseteq [n] \times [n] \times [n]$ hypergraph, independence number $\alpha(H) \leq Q(T)$ for any T that "fits" H. E.g. cap sets, sunflowers, corners, ...

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$$0 \leq Q(T) \leq SR(T) \leq n \leq R(T) \leq n^2$$

$$AR(T)$$

$$GR(T)$$

$$R^G(T)$$

Slice rank

$$T = \sum_{i=1}^{a} \sum_{j} u_i \otimes v_{ij} \otimes w_{ij} + \sum_{i=1}^{b} \sum_{j} u'_{ij} \otimes v'_i \otimes w'_{ij}$$

$$+ \sum_{i=1}^{c} \sum_{j} u''_{ij} \otimes v''_{ij} \otimes w''_i$$

minimize $a+b+c$

$SR(T)$

Geometric rank

$$\operatorname{codim} \{ (u,v) \in \mathbb{F}^n \times \mathbb{F}^n : \forall w \ \ T(u,v,w) = 0 \}$$

$GR(T)$

$\vdots$

$T \in \mathbb{F}^{n \times n \times n}$

Generally :

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$$\begin{array}{lcccccccccc}
\text{Generally:} & 0 & \le & Q(T) & \le & SR(T) & \le & n & \le & R(T) & \le n^2 \\
 & & & & & AR(T) & & & & & \\
 & & & & & GR(T) & & & & & \\
 & & & & & R^{G}(T) & & & & & \\
\text{On random tensors } T: & & & \underbrace{\qquad}_{?} & & \underbrace{\qquad\qquad}_{\approx n} & & & & \underbrace{\qquad\qquad}_{\approx n^2} &
\end{array}$$

3. Subrank of Random Tensors

Theorem For almost all $T \in \mathbb{F}^{n \times n \times n}$ we have $Q(T) = \theta(\sqrt{n})$

Remarks:

- "Almost all" = "random" = generic
- that is, there is a nonempty Zarisk-open $\mathcal{U} \subseteq \mathbb{F}^{n \times n \times n}$ such that for all $T \in \mathcal{U}$ we have $Q(T) = \theta(\sqrt{n})$

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- Previously: $Q(T) \leq n^{2/3 + o(1)}$
- Also for higher-order tensors
- Application: Subrank is not additive under direct sum.

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$Q(n) :=$ subrank of a generic tensor in $\mathbb{F}^{n\times n\times n}$

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Lemma 1 $Q(n) =$ largest r such that $\dim C_r = \underbrace{\dim \mathbb{F}^{n \times n \times n}}_{n^3}$.

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Let $t = Q(n)$

Then $n^3 = \dim C_t \leq n^3 - t(t^2 - 3n + 2)$.

Then $t^2 - 3n + 2 \leq 0$

So $t \leq \sqrt{3n-2}$

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$X_r = \{ \text{tensors in } \mathbb{F}^{n \times n \times n} \text{ with } [r] \times [r] \times [r] \text{ subtensor arbitrary diag.} \}$

$$\Psi_r : GL_n \times GL_n \times GL_n \times X_r \to \mathbb{F}^{n \times n \times n}$$
$$(A, B, C, T) \longmapsto (A \otimes B \otimes C) T \qquad \text{has image } C_r$$

$\square$

Lower bound

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has image C_r

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tangent space of X_r

$$\begin{aligned} (d\psi_r)_{(g_1, g_2, g_3, T)} : \; & Mat_{n \times n} \times Mat_{n \times n} \times Mat_{n \times n} \times Y_r \to \mathbb{F}^{n \times n \times n} \\ & (A, B, C, T) \mapsto ((A \otimes g_2 \otimes g_3) + (g_1 \otimes B \otimes g_3) + (g_1 \otimes g_2 \otimes C)) T \\ & \qquad + (g_1 \otimes g_2 \otimes g_3) S. \end{aligned}$$

Tensor space decompositions

Goal: write tensor space $\mathbb{F}^{n \times n \times n}$ as a sum of tensor subspaces, as efficiently as possible such that each subspace has the form of an $n \times n$ matrix subspace tensored with $\mathbb{F}^n$

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$X \subseteq \mathrm{Mat}_{n \times n} = \mathbb{F}^n \otimes \mathbb{F}^n$

$X[1] = \mathbb{F}^n \otimes X \subseteq \mathbb{F}^{n \times n \times n}$

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$X[3] = X \otimes \mathbb{F}^n$

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Theorem there are subspaces $X_i \subseteq \mathrm{Mat}_{3,3}$ of dim 3 each, such that

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Theorem There are subspaces $X_i \subseteq (\mathbb{F}^n)^{\otimes n-1}$ of dim n^{n-2} each, such that

$$(\mathbb{F}^n)^{\otimes n} = X_1[1] + X_2[2] + \cdots + X_n[n].$$

Again: dimensions match.

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$$\begin{array}{ccc}
X_1 & X_2 & X_3 \\[1em]
\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} &
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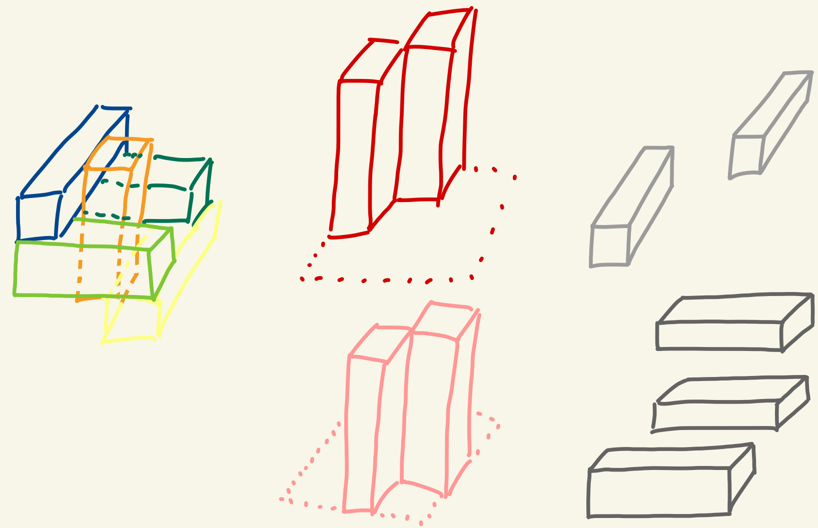

Application: Subrank is not additive under direct sum

Theorem There are tensors $S, T \in \mathbb{F}^{n \times n \times n}$ such that $Q(S), Q(T) \leq \sqrt{3n-2}$ while $Q(S \oplus T) \geq n$.

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Proof idea

- Let T be "random".
- Let $S = I_n - T$. Then S is "random".
- Then $Q(S), Q(T) \leq \sqrt{3n-2}$ by our theorem.
- On the other hand, $Q(S \oplus T) \geq Q(S+T) = Q(I_n) = n$. $\square$

Selected Open Problems

1. Our upper bound $Q(T) \leq \lfloor \sqrt{3n-2} \rfloor$ for generic $T \in \mathbb{F}^{n \times n \times n}$ is tight for $n \leq 100$. Is this always true?

2. Determine all possible tensor space decompositions

3. What is the largest gap between $Q(S \oplus T)$ and $Q(S) + Q(T)$?

New directions: asymptotic gap in the subrank

$S \in V_1 \otimes V_2 \otimes V_3$

$T \in W_1 \otimes W_2 \otimes W_3$

Kronecker product: $S \boxtimes T \in (V_1 \otimes W_1) \otimes (V_2 \otimes W_2) \otimes (V_3 \otimes W_3)$

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Question How does $Q(T^{\boxtimes n})$ behave when $n \to \infty$?

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Theorem (Christandl, Gesmundo, Z) Let $T \in V_1 \otimes V_2 \otimes V_3$ be any tensor.
Exactly one of the following is true:

i) $T = 0$

ii) $Q(T^{\boxtimes n}) = 1$ for all n

iii) $Q(T^{\boxtimes n}) = 1.88^{n - o(n)}$ for all n

iv) $Q(T^{\boxtimes n}) \geq 2^n$ for all n