On the regularity of certain apolar schemes

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AGATES
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Work in progress with A. Oneto and D. Taufer

$F \in S^d \mathbb{C}^n$ homogeneous polynomial

Many ways of decomposing it

$F = \sum_{i=1}^{s} L_i^d$	$F = \sum_{i=1}^{s} L_i^{d-1} M_i$	...	$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i$
Waring decomposition	Tangential decomposition		Gegneralized additive decomposition (GAD)

[IK] Generalized additive decomposition (GAD)

$$\boxed{F = \sum_{i=1}^{s} L_i^{d-k_i} M_i} \qquad 0 \leq k_i \leq d$$

linear form

$L_i \nmid M_i, \qquad L_i \nparallel L_j$

SET UP

Apolarity

$T = \mathbb{K}[y_0, \dots, y_n]$ $\qquad$ $S = \mathbb{K}[x_0, \dots, x_n]$

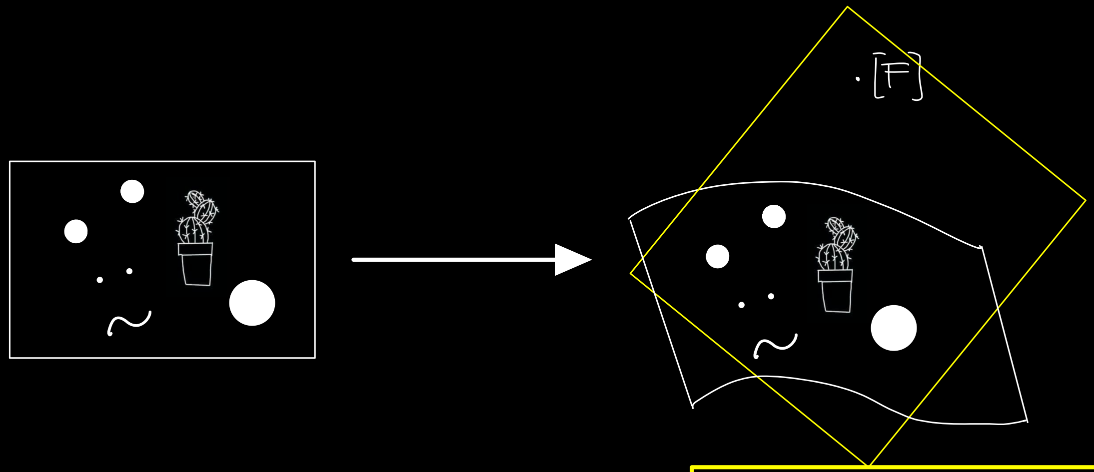
T acts on S by contraction

$$\underline{y}^{\underline{\alpha}} \circ \underline{x}^{[\underline{\beta}]} = \begin{cases} \underline{x}^{[\underline{\beta} - \underline{\alpha}]}, & \beta_i \geq \alpha_i \, \forall i \\ 0 & \text{otherwise} \end{cases}$$

Annihilator of $F \in S_d$

$$F^{\perp} := \{ G \in T_{\leq d} \,/\, G \circ F = 0 \}$$

$$\mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_1) \xrightarrow{\nu_d} \mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$$



Apolarity lemma

$$Z \subset \mathbb{P}(\mathbb{C}^n) \longleftrightarrow \boxed{\begin{array}{c} F \in \langle \nu_d(Z) \rangle \\ \Updownarrow \\ I(Z) \subset F^{\perp} \end{array}}$$

Proposition:

Any apolar scheme Z to F $\left(I(Z) \subseteq F^{\perp}\right)$ contains

$Z' \subseteq Z$ evincing GAD of F

Corollary: If Z is minimal by inclusion among (IRREDUNDANT) the apolar schemes to F $\Rightarrow$ Z evinces a GAD of F

Given Z apolar to F. How to find $Z' \subset Z$ evincing a GAD of F and the corresponding GAD?

Start with your favorite Z apolar to $F \in S_d$

$$F = x^3 + 3x^2y + 3x^2z + 3xy^2 + 12xyz + y^3 + 3y^2z$$

$$I(Z) = I = (y, z^3) \cap (x-y, z)$$

supported at x and $(x+y)$

Find $G \in S_{D \gg d}$ s.t. $G|_d = F$

$$J' = \operatorname{Ker}(\text{Henkel } G_x) \supset I(Z)$$

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supported at x and $(x+y)$

$$J' = (xy^2 - y^3,\ z^2) \supseteq I(Z)$$

$V(J')$ evinces a GAD of $G \in S^6(\mathbb{C})$

(How? $\rightarrow$)

$$G = \frac{1}{120}\left(30x^4yz + (x+y)^5(x+y+6z)\right)$$

$$\partial_x^3 G = \boxed{F = 6xyz + (x+y)^2(x+y+3z)}$$

GAD of F

evinced by $V(J) = Z'$ $\left(I(Z') \supseteq J' \supseteq I(Z)\right)$

Natural way to associate a 0-dim'l scheme to a given GAD

GAD

$f_{x_0} = F\big|_{x_0=1} \in S = K[x_1,\dots,x_n]$

$$F = \sum_{i=1}^{s} L_i^{d-k_i} N_i \quad (*)$$

$L_i^{d-k_i} N_i = F_i$

$F_i \longleftrightarrow Z_{F_i,L_i} : V(f_{L_i}^{\perp})$

$T = K[y_1,\dots,y_n]$ acts on S via apolarity

$T_{f_{x_0}} = T/f_{x_0}^{\perp}$: local, Artinian Gor ring

$Z = \hat{\bigcup_{i=1}} Z_{F_i,L_i}$: Scheme evincing the GAD $(*)$

$I(Z_{F_i,L_i}) \leftarrow T/f_{L_i}^{\perp}$

Ex:

$$F = x_0\,(x_0 - x_2)\,(x_0 + 2x_1 + x_2) = L^{3-2}Q$$

look for apolar scheme supported at

- x_0
- $f_{x_0} = 1 + 2x_1 - 2x_1x_2 - x_2^2$
- $f^{\perp} = (y_1^2, y_1y_2 - y_2^2)$
- lenght 4
  supp (1:0:0)

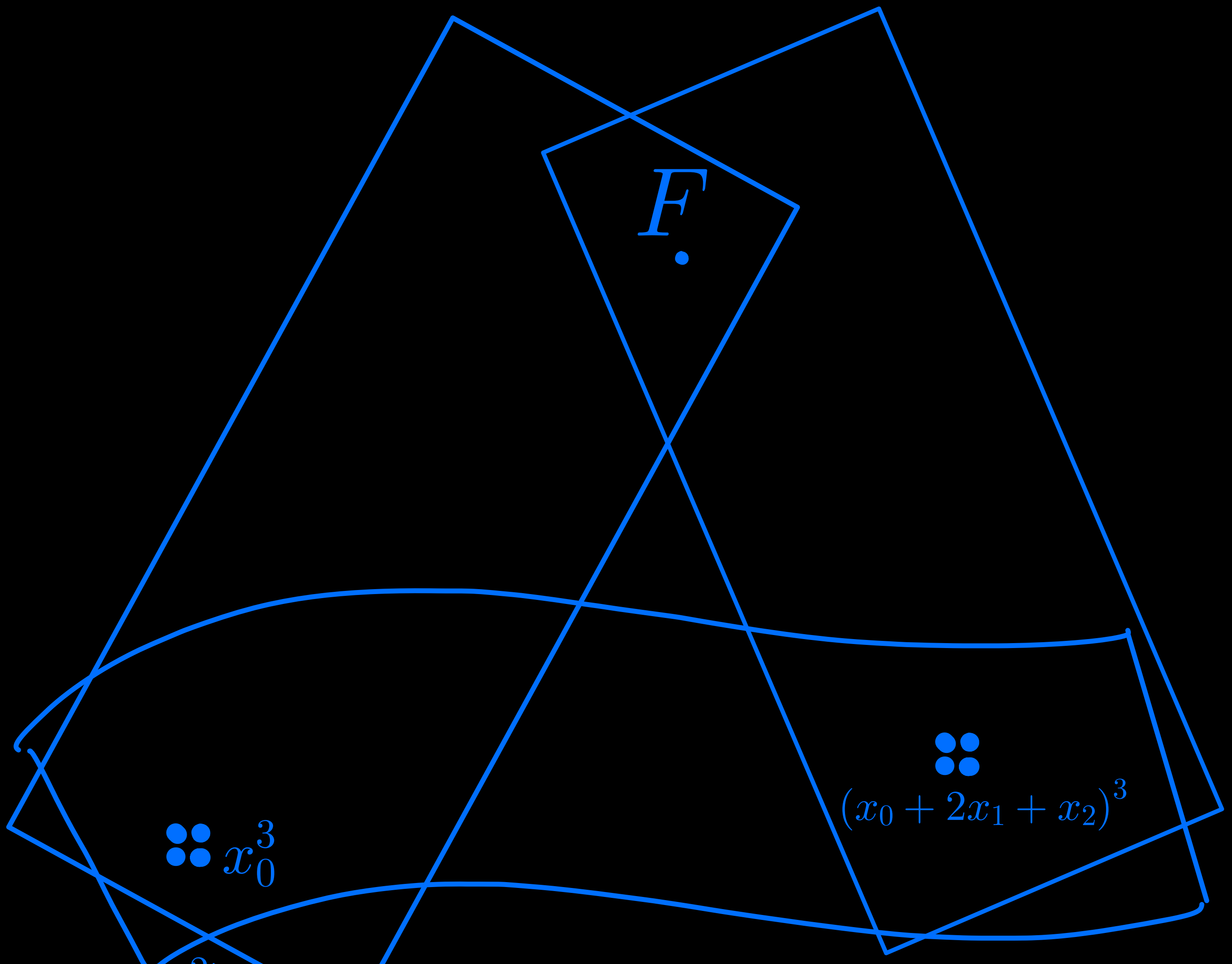
GAD

$$\boxed{F = x_0(x_0^2 + 2x_0x_1 - 2x_1x_2^2)}$$

- $(x_0 + 2x_1 + x_2)$
- $f_{x_0+2x+x_2}$
- $f^{\perp} = (y_1^2 - 4y_1y_2 + 4y_2^2, y_0^2 - 2y_0y_1 + 2y_0y_2 - 3y^2)$
- lenght 4
  supp (1:2:1)

GAD

$$\boxed{F = (x_0 + 2x_1 + x_2)\,(x_0^2 - x_0x_2)}$$



$f^{\perp} = (y_1^2, y_1 y_2 - y_2^2)$

$f^{\perp} = (y_1^2 - 4y_1y_2 + 4y_2^2, y_0^2 - 2y_0y_1 + 2y_0y_2 - 3y^2)$

Our main question:

Are the schemes computing a GAD regular in $\deg(F)$?

$$\bigcup_{i=1}^{r} Z_i = \bigcup_{i=1}^{r} V\left(f^{\perp}_{L_i}\right)$$

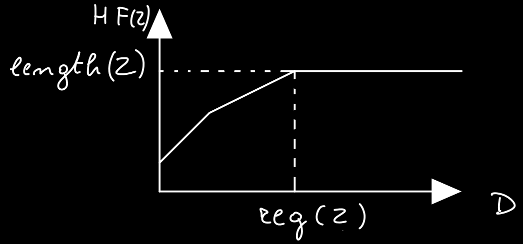
$$\Updownarrow$$

$$F = \sum_{i=1}^{r} L_i^{d-k_i} M_i$$

Z is regular in deg d if
$HF(Z)_d = \text{lenght}(Z)$

$\operatorname{codim} I(Z)_d$

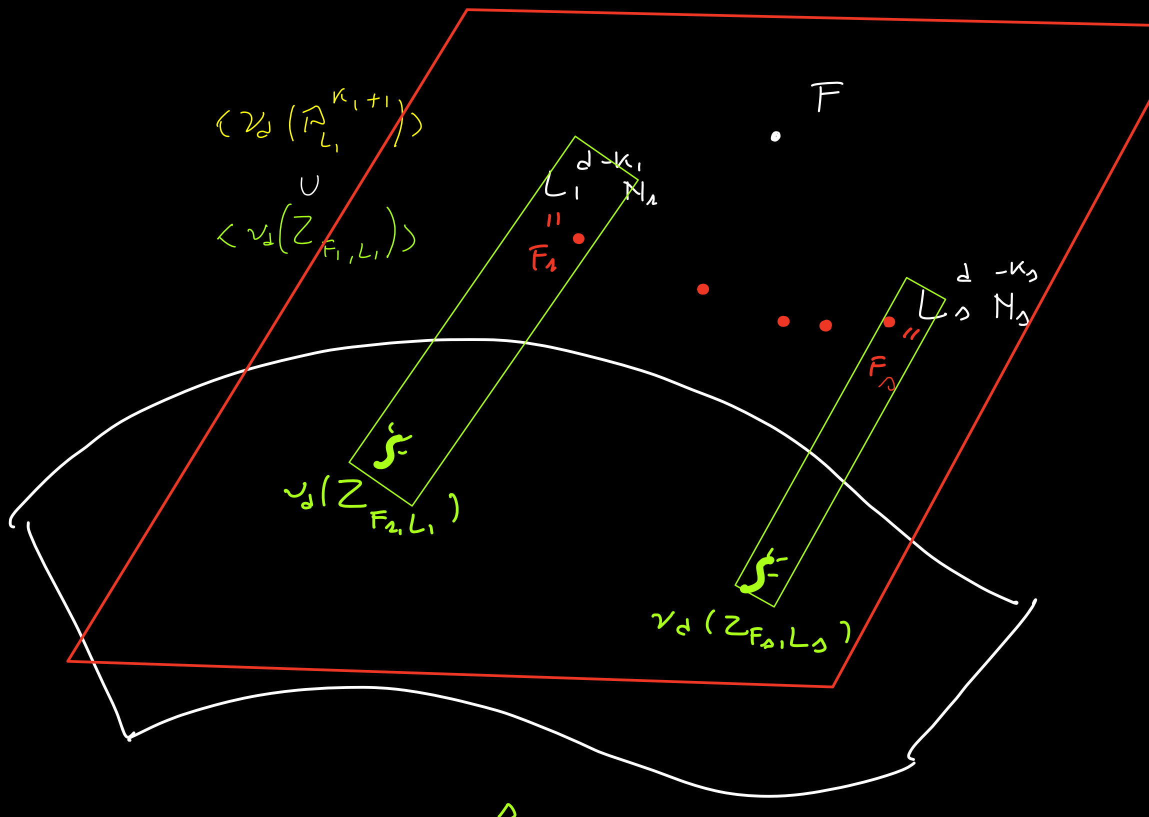
$\dim_K\left(R/I(Z)\right)$



Rephrasing

$$F = \sum_{i=1}^{s} \underbrace{L_i^{d-k_i} M_i}_{= F_i}$$

$$F_i = L_i^{d-k_i} M_i \in \langle \nu_d(Z_{F_i,L_i}) \rangle$$



$$Z = \bigcup_{i=1}^{s} Z_{F_i,L_i}$$

regular in deg d $\iff$ $\dim \langle \nu_d(Z) \rangle = \sum_{i=1}^{s} \dim \langle \nu_d(Z_{F_i,L_i}) \rangle$

$(*)$

- Each Z_{F_i, L_i} is regular in deg d

Because $Z_{F_i, L_i} \subseteq V(P_i^{k_i+1})$ which is regular $\forall\, k_i \leqslant d$

$$\Rightarrow \quad F = L^{d-k} H \qquad 0 \leqslant k \leqslant d$$

$Z_{F, L}$ is reg. in deg d

MINIMALITY

- If $F = \sum_{i=1}^{s} L_i^d$ and s minimal (= Waring rank)

$$\Updownarrow$$

$$\bigcup Z_{L_i^d, L_i} = \bigcup Z(P_i) \text{ regular in deg } d$$

$$I\left(\bigcup Z_i\right) = \bigcup P_i \quad \text{simple pts}$$

- If $F = \sum_{i=1}^{s} L_i^{d-1} M_i$ and <u>s minimal</u> (= tangential decomp.)

$$Z_{L_i^{d-1}M_i,\, L_i} = Z\left(\mathbb{P}_i^2 \cup m_i\right) = 2\text{-jet}$$

$Z = \bigcup Z_i$ regular in degree d

Idea ↓

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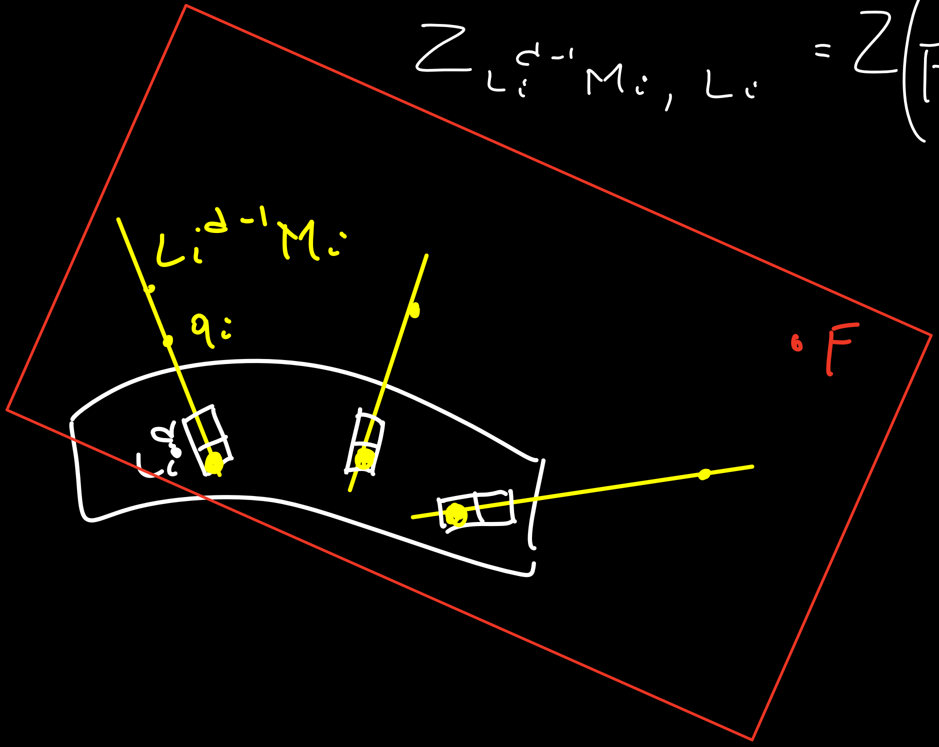
If $F = \sum_{i=1}^{s} L_i^{d-1} M_i$ and s minimal

(= tangential decomp.)

$$Z_{L_i^{d-1}M_i,\, L_i} = Z\left(P_i^2 \cup m_i\right) = 2\text{-jet}$$



used to prove

$\{L_i^{d-1}\}$ l.i. ⇐

$\Rightarrow \{L_i^{d}\}$ l.i.

$\Rightarrow$ none of → can be removed

$\langle \nu(Z)\rangle = \langle L_i^d, q_i \rangle_{i=1,\dots,s}$

↓

these cannot be removed

otherwise $\dim \langle L_1^d \dots L_s^d, q_2, \dots, q_s\rangle = \dim \langle \nu_d(Z)\rangle$

- Analogously $$F = \sum_{i=1}^{s} L_i^{d-k_i} M_i^{k_i}$$ $s \leq n = \#\text{vars}$ and minimal

$$Z_i = Z\left(P_i^{k+1} \cap m_i\right) = (k_i+1)\text{-jet}$$

$$Z = \bigcup_{i=1}^{s \leq n} Z_i \quad \text{regular in degree } d$$

- More generally

$$F = \sum_{i=1}^{s} x_i^{k_i} G_i \quad , \quad \deg G_i \leq k_i + 1 \ , \text{ minimal}$$

$s \leq \#\text{vars}$

$\Rightarrow Z$ is d-regular

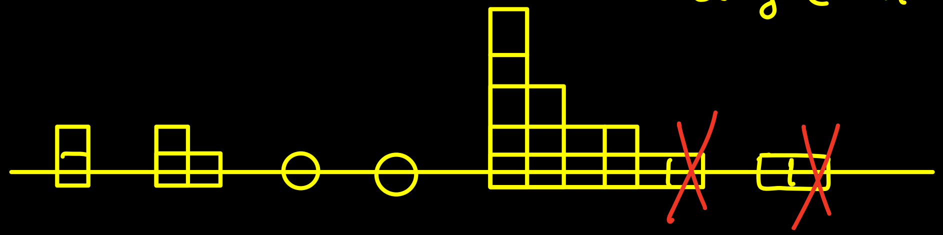
- If $Z \subset F^1$ and $\deg(Z) \leq 2d-1 \Rightarrow$

$\exists\, Z' \subsetneq Z \subset F^1$, Z' reg in deg d

(MINIMALITY)
IRREDUNDANT

([EP. 1988]
[BGI 2010])

A scheme $Z \subset \mathbb{P}^n$, $\deg(Z) \leq 2d-1$
is irregular in deg d $\Leftrightarrow$
$\exists\, \ell \subset \mathbb{P}^n$ line s.t. $\deg(\ell \cap Z) \geq d+2$



Are the schemes computing a irredundant GAD of F regular in deg(F)?

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NO: Irredundant $\not\Rightarrow$ d-regularity

Ex $\quad F = x_0 G_1 + x_1 G_2 \in S^4 \mathbb{C}^3$

$$G_1 = 10x_0^3 - 4x_0^2x_1 + 4x_0^2x_2 - 4x_0x_1^2 - 8x_0x_1x_2 - 3x_0x_2^2 - 8x_1^3 - 4x_2^3$$

$$G_2 = 5x_0^3 + 9x_0x_1^2 - 5x_1^3 - 7x_1^2x_2 + 6x_1x_2^2 - x_2^3$$

$$Z_F = \underset{\text{deg } 6}{Z_1} \cup \underset{\text{deg } 6}{Z_2} \qquad HF(I(Z)) = [1, 3, 6, 10, \boxed{11}, 12 \dots]$$

$6+6 = 12 < 11$

Z not reg. in deg 4

$$I(Z_1) = (-3y_0y_1y_2 + y_1^3, \; y_1^2y_2, \; y_1y_2^2, \; y_1^3 - 2y_2^3)$$

$$I(Z_2) = (y_2^4, \; y_0^3 + 5y_2^3, \; y_0y_2)$$

See the irredundancy →

See the irredundancy

$$F = x_0 G_1 + x_1 G_2$$

If $Z' \subsetneq Z$ apolar to $F \Rightarrow Z'$ evinces a GAD

$$F = x_0^{e_1} Q_1 + x_1^{e_2} Q_2$$

$$(1 \leq e_i \leq 4)$$

(original GAD of F) − (new GAD of F)

$$\Rightarrow x_0 \left(x_0^{e_1 - 1} Q_1 - G_1 \right) + x_1 \left(x_1^{e_2 - 1} Q_2 - G_2 \right) = 0$$

$$\Rightarrow \exists\, T \in S^2 : \quad x_1 T = x_0^{e_1 - 1} Q_1 + G_1 \,, \quad x_0 T = - x_1^{e_2 - 1} Q_2 + G_2$$

$\Rightarrow Z'$ evinces a GAD: $F = x_0 (G_1 + x_1 T) + x_1 (G_2 - x_0 T)$

for some $T \in S_2$

$$Z' \subset Z = Z_1 \cup Z_2$$

$$\Rightarrow Z' = Z'_1 \cup Z'_2 \quad , \quad Z'_i \subseteq Z_i$$

$$\Downarrow$$

$$I(Z_i) \subseteq I(Z'_i)$$

$$\downarrow$$

impose these conditions leads to

$$T = \delta x_0 x_1$$

$\Rightarrow Z' \subset Z$ apolar to F evinces a good!

$$x_0 (G_1 + \delta x_0 x_1^2) + x_1 (G_2 - \delta x_0^2 x_1)$$

Such a Z' turns out to be indep. on δ and $Z' = Z$

$\Rightarrow Z$ irred. #

- $F = x_0 G_1 + x_1 G_2$

  Z irredundant and irregular

  $HF(I(Z)) = [1, 3, 6, 10, \boxed{11}, 12 \dots]$

- a random $I(\tilde{Z}) \subseteq F^\perp$ irred. and supp. of x_0, x_1 is d-regular

- There is a small family of $I(\tilde{Z}) \subseteq F^\perp$ even more irreg. than Z.

Summary

- Simple points : Waring decomposition
- Few points + irredundancy
- Low degree
- Irredundancy

Do there exist infinitely many apolar schemes on prescribed support?

$\Rightarrow$ positive dim'l family of d-reg [BB]?

What about if $\exists!$ apolar scheme on prescribed supp?

Thanks!