Gröbner degenerations of smooth projective curves

Conjecture (Constantinescu, De Negri, _ , 2019)

Let I be a homogeneous prime ideal of a standard graded polynomial ring $S = K[X_0, \ldots, X_n]$ over a field K such that $\operatorname{Proj} S/I$ is a smooth projective curve of genus $g > 0$. Then $\mathrm{in}_<(I)$ is not radical for any monomial order.

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- $K = \mathbb{Q}$ and $g = 1$.

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If $\mathrm{in}_<(I)$ is radical for some $I \subset S$ defining a smooth projective curve of genus g and monomial order $<$, then $\mathrm{in}_<(I)$ should be the Stanley-Reisner ideal of a graph Δ with g cycles.

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Let Δ be a graph without leaves. Then there is no homogeneous ideal I of S defining a smooth curve such that $\text{in}_<(I) = I_\Delta$ for any monomial order $<$.

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What if Δ is a (connected, not a tree) graph with leaves? Try to fix a lex monomial order to start...

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Let Δ be a triangle with a pendant, i.e. the graph on 4 vertices with edges $12, 23, 13, 14$.

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Remark

For any graph with only 1 cycle (possibly with leaves) the answer is no for $K = \mathbb{Q}$ (or K is a totally real algebraic extension of $\mathbb{Q}$), but it would be nice to have an easier proof also in this case.