EXERCISE SHEET

This is the exercise sheet for the mini-course "*Geometry of eigenvectors of tensors*".

1. Singular vector tuples and critical points

- Foundations for various notions of eigenvalues and eigenvectors for tensors: [1, 3, 6, 10, 11, 12, 13]. The exercises below are largely inspired by these references.
- Foundations for the theory of ED degrees: [4].

Exercise 1.1. *With the help of a computer algebra software, use defining equations – rank constraints – to calculate the number of singular vector triples of a general (random) 3-way tensor* $T \in \mathbb{C}^{\mathbf{m}}$ *when* $\mathbf{m} = (2,2,2), \mathbf{m} = (2,2,3), \mathbf{m} = (2,2,4), \mathbf{m} = (3,3,3)$. *How these numbers compare?*

A little Summer game. What is the missing number in the following sequence?

$$10 \quad 11 \quad 12 \quad 13 \quad 20 \quad ? \quad 1000$$

The number of singular vector triples of a general 3-way tensor for $\mathbf{m} = (2,2,3)$ is a hint.

Exercise 1.2. *Using Friedland-Ottaviani's formula, calculate the number of singular vector tuples of a general d-way tensor* $T \in \mathbb{C}^{\mathbf{m}}$ *for* $\mathbf{m} = (2,\dots,2)$.

Exercise 1.3. *Let* $\mathbf{m} = (2,2,3)$ *and* $T = [t_{i,j,k}] \in \mathbb{C}^{\mathbf{m}}$ *be such that* $t_{1,1,1} = 1, t_{2,2,3} = 2$ *and zero otherwise. Compute its singular vector triples.*

Exercise 1.4. *Let* $m \geq 2$, $\mathbf{m} = (m,m,\dots,m)$. *Let* $T = [t_{i_1,\dots,i_d}] \in \mathbb{C}^{\mathbf{m}}$ *be such that* $t_{i,i,\dots,i} = \alpha_i \neq 0$ *for all* $1 \leq i \leq m$ *and zero otherwise. Determine all the E-eigenpairs of* T.

Exercise 1.5. *Let* $d, m \geq 2$ *be even and* $\mathbf{m} = (m,\dots,m)$. *Find a d-way tensor* $T \in \mathbb{C}^{\mathbf{m}}$ *with* no real *E-eigenpairs.*

Exercise 1.6. *Let* $\mathbf{m} = (m,\dots,m)$ *and* $T \in \mathbb{C}^{\mathbf{m}}$ *be a d-way tensor. Show that if* T *has real entries and either* d *or* m *is odd, then* T *has a real eigenpair.*
Hint: *Use* [7, Corollary 13.2] *and the observation that* $\mathbb{R}$ *is a 2-field in the sense explained just before the referred statement in loc. cit.*

Exercise 1.7. *Let* $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ *be the unit sphere and let* $f \in S^d\mathbb{R}^m$ *be a symmetric tensor.*

(a) *What are the E-eigenvectors of* f *whose corresponding eigenvalue is zero?*
(b) *Show that a vector* $v \in \mathbb{S}^{m-1}$ *is an E-eigenvector of* f *if and only if* v *is a critical point of the function* f *naturally defined on* $\mathbb{S}^{m-1}$. *(Use Lagrange multipliers.)*
(c) *What is the eigenvalue corresponding to an E-eigenvector* $v \in \mathbb{S}^{m-1}$ *of* f*?*

Exercise 1.8. *Let* $f(x) = x_1x_2x_3(x_1+x_2+x_3)$. *Compute its E-eigenvectors and check that all of them are real. (Khozasov* [9] *proved that for any* d *and* m *there exist real homogeneous polynomials with the property of having* $\frac{(d-1)^m-1}{d-2}$ *real E-eigenvectors. His construction is based on* harmonic polynomials.)

Exercise 1.9. *Find a nonzero complex homogeneous polynomial* f *such that every normalized vector* v*, i.e.* $v^tv = 1$, *is an E-eigenvector of* f.

Exercise 1.10. *Let* $\lambda \neq 0$ *and* $d \geq 3$, *and let* $f(x)$ *be a complex homogenous polynomial in* m *variables of degree* d. *Then* $v \in \mathbb{C}^m$ *is a normalized E-eigenvector of* f *(i.e.* $v^tv = 1$*) with eigenvalue* λ *if and only if* v *is a singular point of the affine hypersurface defined by*

$$f(x) - \frac{\lambda}{2}x^t \cdot x - \left(\frac{1}{d} - \frac{1}{2}\right)\lambda = 0.$$

Exercise 1.11. *Choose your favorite positive semidefinite real homogeneous polynomial* f *(i.e.* $f(x) \geq 0$ *for all* $x \in \mathbb{R}^m$*) of degree* $d \geq 4$ *and in* $m \geq 3$ *variables. Calculate its E-eigenpairs and verify Qi's result* [13, Theorem 5(b)].

Exercise 1.12. *Let $p_1, \dots, p_d$ be d distinct points on $\mathbb{P}^1$. Is there a d-way tensor $T \in \mathbb{C}^{(2,\dots,2)}$ such that its E-eigenvectors are the p_i's? Is there a symmetric tensor $f \in S^d\mathbb{C}^2$ such that its E-eigenvectors are the p_i's? (See* [1, Theorem 2.7].*)*

Exercise 1.13. *Let $\mathcal{M}_r$ be the variety of real $m \times n$ matrices with $m \leq n$ of rank $\leq r$. For a general real $m \times n$ matrix A its SVD is $A = U\Sigma V^t$ where $\Sigma = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$ is pseudo-diagonal with $\lambda_1 > \dots > \lambda_m > 0$, and U, V are orthogonal matrices.*
Using Terracini's lemma to describe the tangent space to $\mathcal{M}_r$, show that, for the Frobenius distance function from A to the smooth variety $\mathcal{M}_r \setminus \mathcal{M}_{r-1}$, the matrices

$$C_{i_1,\dots,i_r} = U\mathrm{diag}(\lambda_{i_1}, \dots, \lambda_{i_r})V^t$$

are critical points. (One can prove that these are all: in terms of ED degrees [4]*, this says that* $\mathrm{EDdegree}(\mathcal{M}_r) = \binom{m}{r}$.*)*

2. Chern classes

- Some of the main references for intersection theory in algebraic geometry (Chern classes, Chern polynomials, and much more): [5, 7, 8]; see [2, Chapter IV] for a topological introduction.

In the following, by a *variety* we mean an irreducible and reduced separated scheme of finite type over an algebraically closed field of characteristic zero (e.g. a smooth projective variety over that field).

The *splitting principle* for vector bundles is a very useful tool to compute Chern classes. It is the content of the following result: it roughly says that every relation among Chern classes can be established assuming that the bundles we are working with are split.

Theorem. *Let X be a smooth variety and $\mathcal{E}$ a vector bundle of rank r on X. Then there exists a smooth variety Y and a morphism $\varphi : Y \to X$ such that the pullback $\varphi^*\mathcal{E}$ has a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_r = \varphi^*\mathcal{E}$ where $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is a line bundle and the pullback map $\varphi^* : A(X) \to A(Y)$ is an injective ring homomorphism. Moreover, $c_{\varphi^*\mathcal{E}} = \prod_{i=1}^r c_{\mathcal{L}_i}$, i.e. $\varphi^*\mathcal{E}$ has the same Chern classes of $\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_r$.*

Exercise 2.1. *Using the splitting principle and that $c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L})$ for any line bundle $\mathcal{L}$ on X, check that*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E}).$$

Exercise 2.2. *Let X be a smooth variety. Let $\mathcal{L}$ and $\mathcal{E}$ be a line bundle and a vector bundle of rank r on X, respectively. Using the splitting principle, find a formula for the kth Chern class $c_k(\mathcal{E} \otimes \mathcal{L})$ in terms of Chern classes of $\mathcal{E}$ and $\mathcal{L}$. (When $r = 1$, $c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + c_1(\mathcal{L})$.)*

Exercise 2.3. *If $\mathcal{E}$ is vector bundle of rank r on X, factor its Chern polynomial $c(t, \mathcal{E}) = \prod_{j=1}^r (1 + \xi_j(\mathcal{E})t)$, where $\xi_j(\mathcal{E})$ are the Chern roots. The Chern character of $\mathcal{E}$ is $ch(\mathcal{E}) = \sum_{j=1}^r e^{\xi_j(\mathcal{E})}$.*
Keep the assumptions from **Exercise 2.2**. *Let $ch(\mathcal{E})$ and $ch(\mathcal{L})$ be the Chern characters of $\mathcal{E}$ and $\mathcal{L}$. Using the splitting principle, prove that*

$$ch(\mathcal{E} \otimes \mathcal{L}) = ch(\mathcal{E}) \cdot ch(\mathcal{L}).$$

Exercise 2.4. *Calculate the Chern polynomial of $\Omega^1_{\mathbb{P}^n}(2)$ where $\Omega^1_{\mathbb{P}^n}$ is the cotangent bundle on $\mathbb{P}^n$. Show that $c_n(\Omega^1_{\mathbb{P}^n}(2)) = 0$ if and only if n is odd.*

Exercise 2.5. *Let X be a smooth variety of dimension n. Suppose the tangent bundle $\mathcal{T}_X$ splits as $\mathcal{T}_X = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_n$, where the $\mathcal{L}_i$ are line bundles with $a_i = c_1(\mathcal{L}_i)$. Show that the canonical bundle of X is isomorphic to $\mathcal{O}_X(-a_1 - \dots - a_n)$. Check that $c_1(\mathcal{T}_X)$ is the class of the anticanonical divisor $-K_X$ indeed. (This last statement is always true.)*

We continue with an important theorem.

Theorem. *Let X be a smooth complex variety of dimension n. Then the degree of $c_n(\mathcal{T}_X)$, i.e. the top Chern number of the tangent bundle $\mathcal{T}_X$ (usually denoted $\int_X c_n(T_X)$) equals its topological Euler characteristic $\chi_{\mathrm{top}}(X)$.*

Exercise 2.6. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree d. Use the tangent exact sequence*

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^{n+1}|X} \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

to find the topological Euler characteristic of X. What does the formula read for a degree d smooth curve in $\mathbb{P}^2$?

Exercise 2.7. *Let $\mathcal{E}$ be a vector bundle of rank r on a variety X. Let $\sigma_0, \ldots, \sigma_{r-i}$ be general elements of a vector space $W \subset H^0(\mathcal{E})$ of global sections generating $\mathcal{E}$. Prove that, when non-empty, the degeneracy locus $D = V(\sigma_0 \wedge \cdots \wedge \sigma_{r-i})$ is generically reduced of codimension i in X.*

Hints:

(a) *Let $\dim W = m$. Define a morphism φ from X to the Grassmannian of $(m-r)$-dimensional linear subspaces of W.*

(b) *Let U be a given subspace of dimension $r-i+1$ generated by global sections $\sigma_0, \ldots, \sigma_{r-i}$. The idea is to define, for a general such U, a locus $\Sigma \subset \mathbb{G}(m-r, W)$ of codimension i such that $\varphi^{-1}(\Sigma) = D$. Kleiman's transversality theorem in characteristic zero* [5, Theorem 1.7] *applied to the Grassmannian $\mathbb{G}(m-r, W)$ shows that D has pure codimension* $\operatorname{codim}(D) = \operatorname{codim}(\Sigma) = i$ *and it is generically reduced. See also the proof of* [5, Lemma 5.2].

Exercise 2.8. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on a variety X. Then its top Chern class $c_r(\mathcal{E}) = 0$ if and only if the zero locus of a general global section of $\mathcal{E}$ is empty.*
The statement is false when one drops the assumption on global generation: let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_X(-a, 0) \oplus \mathcal{O}_X(a, 0)$ for some $a > 0$. Show that $\mathcal{E}$ has vanishing top Chern class but its general global section has nonempty zero locus.

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