Parallel syzygies of higher secant varieties

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Geometry of secants

- $\Bbbk$: an algebraically closed field of characteristic 0.
- $X \subseteq \mathbb{P}^r$: a projective (irreducible and reduced) variety over $\Bbbk$ with $\langle X \rangle = \mathbb{P}^r$.
- $q \in \mathbb{Z}$: a positive integer.
- $\sigma_q(X)$: the q-secant variety to $X \subseteq \mathbb{P}^r$, that is,

$$\sigma_q(X) = \overline{\bigcup \langle x_1, \ldots, x_q \rangle} \subseteq \mathbb{P}^r.$$

- $e = \operatorname{codim} \sigma_q(X)$: the codimension of $\sigma_q(X) \subseteq \mathbb{P}^r$.
- $I(X)$: the homogeneous ideal of $X \subseteq \mathbb{P}^r$.
- $S(X)$: the homogeneous coordinate ring of $X \subseteq \mathbb{P}^r$.

Syzygies

The *graded Betti numbers* $\beta_{i,j}(S(X))$ of $S(X)$ are defined as follows: Consider the minimal free resolution of $S(X)$:

$$0 \longleftarrow S(X) \longleftarrow F_0 \longleftarrow \cdots \longleftarrow F_i \longleftarrow \cdots$$

Then one can write

$$F_i = \bigoplus_{j \in \mathbb{Z}} S^{\beta_{i,j}(S(X))}(-i-j), \quad S = S(\mathbb{P}^r) = \Bbbk[x_0, \ldots, x_r].$$

The *Betti table* of $S(X)$ is a diagram given by

	i	$i+1$
j	$\beta_{i,j}(S(X))$	$\beta_{i+1,j}(S(X))$
$j+1$	$\beta_{i,j+1}(S(X))$	$\beta_{i+1,j+1}(S(X))$

(Images of the F_i are called *syzygy* modules of $S(X)$.)

Syzygies

One says that $S(X)$ satisfies *property $N_{d,p}$* if $\beta_{i,j}(M) = 0$ for all

$$i \le p \quad \text{and} \quad j \ge d.$$

We define the (Castelnuovo-Mumford) *regularity* of $X \subseteq \mathbb{P}^r$ to be

$$\operatorname{reg} X = \min\{d \in \mathbb{Z} : S(X) \text{ satisfies property } N_{d,p} \text{ for all } p \ge 0\}.$$

	0	1	$\cdots$	p	$p+1$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$d-1$	?	?	$\cdots$	?	?	$\cdots$
d	0	0	$\cdots$	0	?	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$\operatorname{reg} X$	0	0	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	

Minimal degree and del Pezzo higher secant varieties

[Ciliberto-Russo 2006] gave key hints on relations among q-secant varieties for q varying. One is that we have

$$\mathbb{C}_z\sigma_q(X) \supset \mathrm{Join}(\mathbb{T}_z\sigma_j(X), \sigma_{q-j}(X)) \quad \text{(irred. compo.)}$$

for a general point $z \in \sigma_j(X)$, $j \leq q$, where

- $\mathbb{T}_z$: the projective tangent space at z, and
- $\mathbb{C}_z$: the projective tangent cone at z.

This fact implies that the degree of $\sigma_q(X)$ is bounded above by the *minimal higher secant degree*:

$$\deg \sigma_q(X) \geq \binom{e+q}{q}.$$

(Recall that $e = \operatorname{codim} \sigma_q(X)$.)

Minimal degree and del Pezzo higher secant varieties

The inequality above is sharp for each $e \geq 0$ and each $q \geq 1$.

Definition

A q-secant variety $\sigma_q(X)$ is *of minimal degree* if

$$\deg \sigma_q(X) = \binom{e+q}{q}.$$

For the classical case $q = 1$, if $\deg X = \operatorname{codim} X + 1$, then $X \subseteq \mathbb{P}^r$ is a *variety of minimal degree*.

Minimal degree and del Pezzo higher secant varieties

By del Pezzo and Bertini, a variety of minimal degree is

1. ($e = 0$) the whole space $\mathbb{P}^r$;
2. ($e = 1$) a quadric hypersurface;
3. ($e = 3$) a cone over the 2-Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$; or
4. ($e \geq 0$) a rational normal scroll $S(a_1, \ldots, a_n)$, $a_i \geq 0$.

A q-secant variety $\sigma_q(X)$ has minimal degree for the following cases:

1. X is a variety of minimal degree.
2. X is, with $q \geq 2$, a smooth del Pezzo surface

$$(\mathbb{P}^2, |3H - \xi|) \quad \text{or} \quad (\mathbb{P}^1 \times \mathbb{P}^1, |O(2,2)|).$$

Refer to [Ciliberto-Russo 2006] for more examples.

Minimal degree and del Pezzo higher secant varieties

By [C.-Kwak 2022], there are no q-secant varieties $\sigma_q(X)$ of degree

$$\binom{e+q}{q} < \deg \sigma_q(X) < \binom{e+q}{q} + \binom{e+q-1}{q-1},$$

and if $\deg \sigma_q(X) = \binom{e+q}{q} + \binom{e+q-1}{q-1}$, then the *sectional genus* of $\sigma_q(X)$ is

$$\pi(\sigma_q(X)) := p_a(\mathbb{P}^{e+1}_{\text{gen.}} \cap \sigma_q(X)) \leq (q-1)\deg \sigma_q(X) + 1.$$

Definition

A q-secant variety $\sigma_q(X)$ is *del Pezzo* if

$$\deg \sigma_q(X) = \binom{e+q}{q} + \binom{e+q-1}{q-1} \text{ \& } \pi(\sigma_q(X)) = (q-1)\deg \sigma_q(X) + 1.$$

Minimal degree and del Pezzo higher secant varieties

For the classical case $q = 1$, the following are equivalent:

1. $X \subseteq \mathbb{P}^r$ is del Pezzo, that is, $\deg X = \operatorname{codim} X + 2$, and $\pi(X) = 1$.
2. $\deg X = \operatorname{codim} X + 2$, and $S(X)$ is Cohen-Macaulay.

Del Pezzo varieties generalize smooth del Pezzo surfaces and have been partially classified.

For del Pezzo higher secant varieties, not many examples are known. Basic examples are

1. elliptic normal curves ([Bothmer-Hulek 2004] or [Fisher 2006]), and
2. the Plücker embedding of $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^{2q+2})$ for each $q \geq 1$.

Minimal degree and del Pezzo higher secant varieties

Higher secant varieties of minimal degree admit a generalization of the following:

Theorem (Han-Kwak 2015)

Suppose that $e = \operatorname{codim} X \geq 1$. *The following are equivalent:*

1. $X \subset \mathbb{P}^r$ *is a variety of minimal degree.*
2. $\beta_{e,1}(S(X)) \neq 0$.
3. $\beta_{p,1}(S(X)) = p\binom{e+1}{p+1}$ *for some/all* $1 \leq p \leq e$.
4. $S(X)$ *satisfies property* $N_{2,p}$ *for some/all* $p \geq e$.

	0	1	$\cdots$	e
0	1	0	$\cdots$	0
1	0	$\beta_{1,1}$	$\cdots$	$\beta_{e,1}$

Minimal degree and del Pezzo higher secant varieties

For simplicity, the following particularly holds:

Theorem (C.-Kwak 2022)

A q-secant variety $\sigma_q(X)$ has minimal degree if and only if $S(\sigma_q(X))$ has Betti table

$$\begin{array}{c|cccc} & 0 & 1 & \cdots & e \\ \hline 0 & 1 & 0 & \cdots & 0 \\ q & 0 & \beta_{1,q} & \cdots & \beta_{e,q} \end{array},$$

where

$$\beta_{p,q} = \binom{p+q-1}{q}\binom{e+q}{p+q}.$$

Minimal degree and del Pezzo higher secant varieties

Moreover, when $\sigma_q(X)$ is of minimal degree with $e \geq 2$, one of the following holds:

1. There is an $(e+q) \times (q+1)$ matrix M of linear forms on $\mathbb{P}^r$ such that
$$I(\sigma_q(X)) = I_{q+1}(M),$$
hence $S(\sigma_q(X))$ is minimally resolved by the Eagon-Northcott complex associated to M.

2. There is a $(q+2) \times (q+2)$ symmetric matrix M of linear forms on $\mathbb{P}^r$ such that
$$I(\sigma_q(X)) = I_{q+1}(M),$$
hence $S(\sigma_q(X))$ is minimally resolved by the Józefiak complex [Józefiak 1978] associated to M.

Minimal degree and del Pezzo higher secant varieties

If X is smooth, then the matrix M of linear forms is given by a suitable decomposition.

1. A multiplication map of the form

$$H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) \to H^0(X, L)$$

give the matrix M of linear forms, where $L = O_X(1)$, the L_i are line bundles on X with $L_1 \otimes L_2 = L(-B)$ for an effective divisor $B \subset X$.

2. A multiplication map of the form

$$H^0(X, L_1) \otimes H^0(X, L_1) \to H^0(X, L_1^2) \to H^0(X, L)$$

give the matrix M of linear forms, where $L = O_X(1)$, L_1 is a line bundle on X with $L_1^2 = L(-B)$ for an effective divisor $B \subset X$.

Minimal degree and del Pezzo higher secant varieties

Similarly:

Theorem (Han-Kwak 2015)

Suppose that $e = \operatorname{codim} X \geq 2$. *The following are equivalent:*

1. $X \subset \mathbb{P}^r$ *is a del Pezzo variety.*
2. $\beta_{p,1}(S(X)) = p\binom{e+1}{p+1} - \binom{e}{p-1}$ *for some/all* $1 \leq p \leq e-1$.
3. $S(X)$ *satisfies property* $N_{2,e-1}$.

	0	1	$\cdots$	$e-1$	e
0	1	0	$\cdots$	0	0
1	0	$\beta_{1,1}$	$\cdots$	$\beta_{e-1,1}$	0
2	0	0	$\cdots$	0	1

Minimal degree and del Pezzo higher secant varieties

Theorem (C.-Kwak 2022)

A q-secant variety $\sigma_q(X)$ is del Pezzo if and only if $S(\sigma_q(X))$ has Betti table

$$\begin{array}{c|ccccc} & 0 & 1 & \cdots & e-1 & e \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 \\ q & 0 & \beta_{1,q} & \cdots & \beta_{e-1,q} & 0 \\ 2q & 0 & 0 & \cdots & 0 & 1 \end{array},$$

where

$$\beta_{p,q} = \binom{p+q-1}{q}\binom{e+q}{p+q} - \binom{e+q-p-1}{q-1}\binom{e+q-1}{p-1}.$$

The Betti table above is symmetric, that is, $S(\sigma_q(X))$ is Gorenstein when $\sigma_q(X)$ is del Pezzo.

Minimal degree and del Pezzo higher secant varieties

Consider a *general inner projection* X_z and a *general tangential projection* $X_{\mathbb{T}_z X}$ of X with $z \in X$, and recall $e = \operatorname{codim} \sigma_q(X)$.

1. $\sigma_q(X)$: min. deg. $\Longrightarrow \sigma_q(X_z)$: min. deg. when $e \geq 1$ and $q \geq 1$.
2. $\sigma_q(X)$: min. deg. $\Longleftarrow \sigma_q(X_z)$: min. deg. when $e \geq 2$ and $q \geq 1$.

1. $\sigma_q(X)$: min. deg. $\Longrightarrow \sigma_{q-1}(X_{\mathbb{T}_z X})$: min. deg. when $e \geq 0$ and $q \geq 2$.
2. $\sigma_q(X)$: min. deg. $\Longleftarrow \sigma_{q-1}(X_{\mathbb{T}_z X})$: min. deg. when $e \geq 0$ and $q \geq 2$.

Minimal degree and del Pezzo higher secant varieties

1. $\sigma_q(X)$: del Pezzo $\Longrightarrow \sigma_q(X_z)$: del Pezzo when $e \geq 2$ and $q \geq 1$.
2. $\sigma_q(X)$: del Pezzo $\Longleftarrow \sigma_q(X_z)$: del Pezzo when $e \geq 3$ and $q \geq 1$.

1. $\sigma_q(X)$: del Pezzo $\Longrightarrow \sigma_{q-1}(X_{\mathbb{T}_z X})$: del Pezzo when $e \geq 1$ and $q \geq 2$.
2. $\sigma_q(X)$: del Pezzo $\Longleftarrow \sigma_{q-1}(X_{\mathbb{T}_z X})$: del Pezzo when $e \geq 2$ and $q \geq 3$.

A generalized $(2g + 1 + p)$-theorem

Consider a complete embedding $C \subseteq \mathbb{P}H^0(C, L) = \mathbb{P}^r$ of a smooth curve with $d = \deg L$.

1. [Mumford 1969] If $d \geq 2g + 1$, then $S(C)$ is Cohen-Macaulay.
2. [Saint-Donat 1972] If $d \geq 2g + 2$, then $I(C)$ is generated by quadratic forms.

Theorem (Green 1984)

Let L be a very ample line bundle of degree d on a smooth curve C of genus g. If

$$d \geq 2g + 1 + p, \quad p \geq 0,$$

then for the complete embedding $C \subseteq \mathbb{P}H^0(C, L)$

1. *$S(C)$ is Cohen-Macaulay, and*
2. *$S(C)$ satisfies property $N_{2,p}$.*

A generalized $(2g+1+p)$-theorem

Theorem (Ein-Niu-Park 2022)

Let L be a very ample line bundle of degree d on a smooth curve C of genus g. Suppose that

$$d \geq 2g + 2q + p - 1, \quad p \geq 0.$$

Then, for the complete embedding $C \subseteq \mathbb{P}H^0(C, L)$,

1. *$S(\sigma_q(C))$ is Cohen-Macaulay,*
2. *$S(\sigma_q(C))$ satisfies property $N_{q+1,p}$,*
3. $\operatorname{reg} \sigma_q(C) = \begin{cases} q+1 & \text{if } g = 0 \\ 2q+1 & \text{if } g > 0 \end{cases}$, *and*
4. $\dim H^0(\sigma_q(C), \omega_{\sigma_q(C)}) = \beta_{e,2q}(S(\sigma_q(C))) = \binom{g+q-1}{q}$.

A generalized $(2g+1+p)$-theorem

For the case $g \geq 1$,

- $S(\sigma_q(X))$ satisfies property $N_{q+1,p}$, and
- $\operatorname{reg} \sigma_q(X) = 2q+1$.

	0	1	$\cdots$	p	$p+1$	$\cdots$
0	1	0	$\cdots$	0	0	$\cdots$
q	0	$*$	$\cdots$	$*$	?	$\cdots$
$q+1$	0	0	$\cdots$	0	?	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$2q$	0	0	$\cdots$	0	?	$\cdots$
$2q+1$	0	0	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	

Ongoing: a generalized gonality conjecture

Consider the *gonality* of a smooth curve C, namely

$$\begin{aligned} \operatorname{gon} C &= \min\{\deg\phi : \phi \text{ maps } C \text{ onto } \mathbb{P}^1\} \\ &= \min\{\delta : C \text{ carries a } g^1_\delta\}. \end{aligned}$$

Green and Lazarsfeld conjectured the following:

Theorem (Ein-Lazarsfeld 2015, Rathmann 2016)

Let L be a very ample line bundle on a smooth curve C, and let ω_C be the canonical line bundle on C. Assume that

$$H^1(C, L) = H^1(C, L \otimes \omega_C^{-1}) = 0 \quad \text{(Rathmann)}.$$

Then, for the complete embedding $C \subseteq \mathbb{P}H^0(C, L) = \mathbb{P}^r$,

$$\beta_{p,1}(S(C)) \neq 0 \quad \textit{if and only if} \quad 1 \leq p \leq r - \operatorname{gon} C.$$

Ongoing: a generalized gonality conjecture

The gonality, minus one, can be generalized as follows:

Definition

We define

$$\gamma^q(C) = \min\{\delta - q : C \text{ carries a } g^q_\delta\}.$$

The *gonality sequence* of C consists of $\gamma^q(C) + q$.

As for the case $q = 1$, any linear system $|L_1|$ of dimension q induces an interesting multiplication map

$$H^0(C, L_1) \otimes H^0(C, L \otimes L_1^{-1}) \to H^0(C, L).$$

Ongoing: a generalized gonality conjecture

Conjecture (see [Sidman-Vermeire 2009] for $\Longleftarrow$)

For any very ample line bundle L of degree $d \gg 0$ on C, we have

$$\beta_{p,q}(S(\sigma_q(C))) \neq 0 \quad \textit{if and only if} \quad 1 \leq p \leq e - \gamma^q(C).$$

It seems that by Ein-Niu-Park's argument and the Serre-Fujita vanishing theorem, there exists an integer $d_0 = d_0(C, q)$ such that

$$\text{the direction} \implies \text{holds whenever } d \geq d_0.$$

Problem

Find an effective lower bound for deg L (or Rathmann type condition) that establishes the conjecture.

Thank you for your attention!