ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

for Zaky

# Fat points are maddening? Try to look at superfat ones (and cuckoo varieties)

Joint work (in progress) with S.Canino, M.V.Catalisano, M.Idà

AGATES – Warsaw, October 2022

## 1. On some 0-dimensional schemes

- We will work in the Wilderness of 0-dim schemes, but we will avoid the Wilderness of the praries such as:

## 1. On some 0-dimensional schemes

- We will work in the Wilderness of 0-dim schemes, but we will avoid the Wilderness of the praries such as:

## 1. On some 0-dimensional schemes

Our wilderness will be (No Cactuses even if we are in Poland!):

**1. On some 0-dimensional schemes in the plane**

- Our wilderness will be like this:

i.e. We will work in a smooth (smoothable) enviroment: the plane!

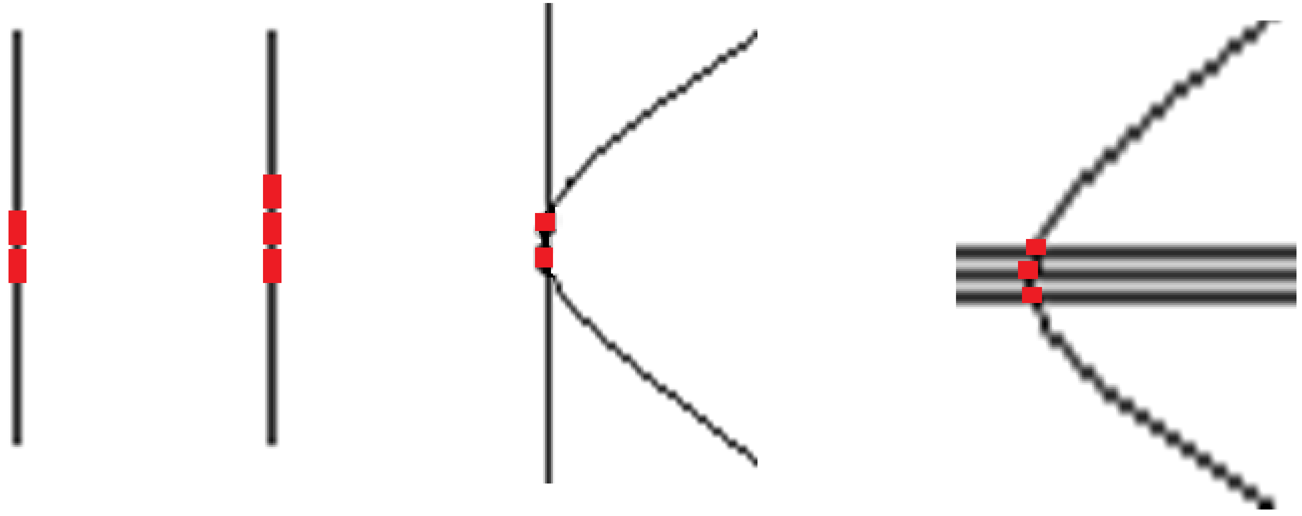
## 1. On some 0-dimensional schemes in the plane

We want to consider how do you visualize a 0-dimensional scheme $Z$ .

Let $Z$ with support at one point; e.g. the *curvilinear points*:

Ideals: $(x,y^2)$ ; $(x,y^3)$ ; $(x-y^2, y^2)$ ; $(x-y^2, y^3)$

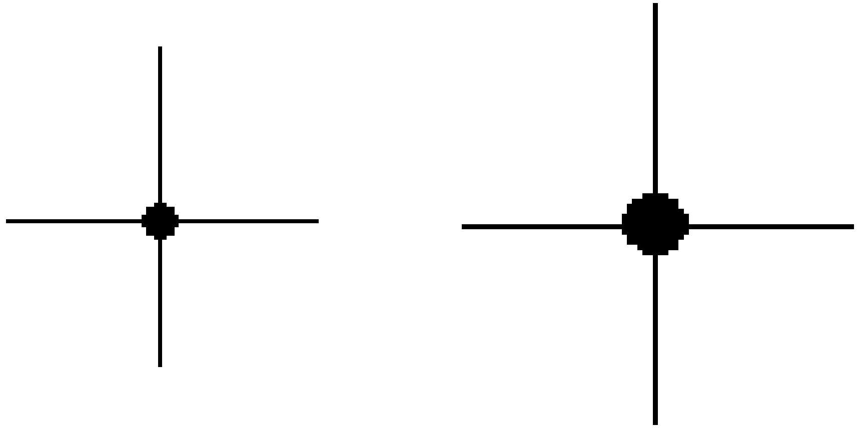
Schemes:



How to visualize a *Fat Point*: $mP$ , of ideal $(I_P)^m = (x,y)^m$ ?

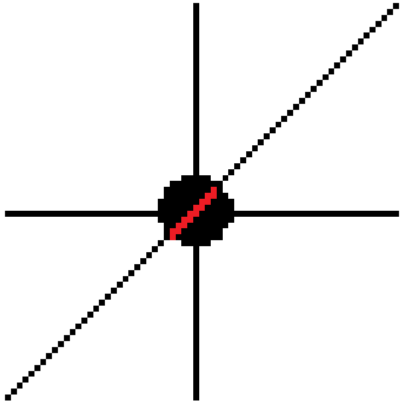
## 1. On some 0-dimensional schemes in the plane

One first image of a fat point is like this:



… and so on

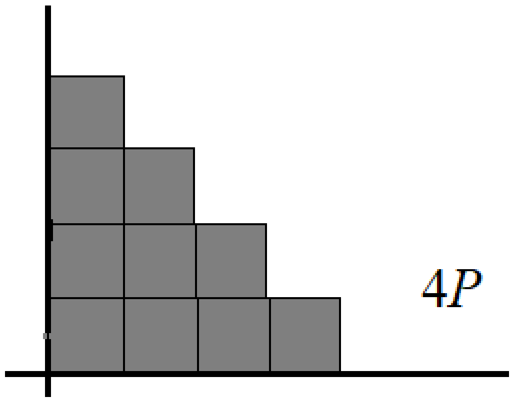
This image is also related to the fact that any line passing through the point *mP* meets the fat point in an *m*-curvilinear point:



In this sense the fat point is *symmetric* , with rispect to any direction: any line intersects the fat point *mP* giving a curvilinear point of lenght *m*.

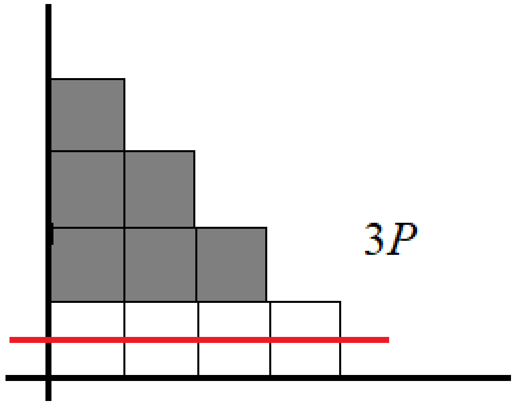
# 1. On some 0-dimensional schemes in the plane

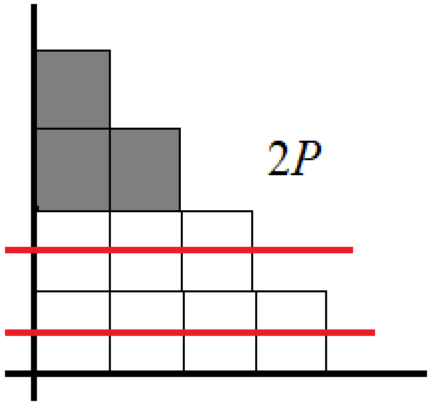
Another image is a triangular one:




This is suggested by Algebra (the monomials in $x,y$), it is a *Pithagorean*, in the sense that it respects the fact that $lenght(mP) = \binom{m+2}{2}$, a triangular number,

and it also respects the behaviour of the residues with respect to lines:

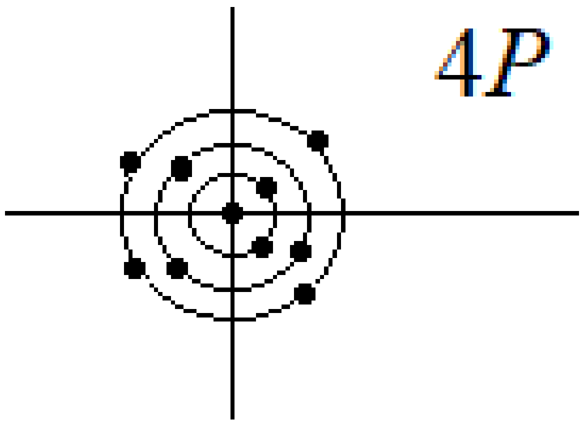



But this image is not round, it is not symmetric.

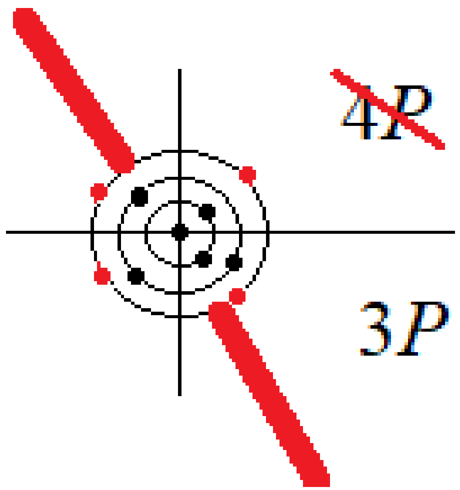
## 1. On some 0-dimensional schemes in the plane

We can put together the two visualizations by viewing the fat point like a *Bohr atom*:




In this image, the $\binom{m+2}{2}$ = 1+2+3+…+*m* *points* which make up the *m*-fat point are thought on «*orbitals*» (classically known as *infinitesimal neighborhoods*), not in definite positions there, but «delocalized along the orbital».

With this image, the residuation works like this:



**1. On some 0-dimensional schemes in the plane**

Let us consider one property of fat points that lead us to give the following definition:

**Definition**: Let $Z \subseteq \mathbb{A}^2$ be a 0-dimensional scheme supported at one point $P$. We will say that $Z$ is $m$-symmetric if for every line $L$ through $P$ we have *length* $(Z \cap L) = m$.

**Remark**: The $m$-fat points $mP$ are the *smallest* $m$-symmetric schemes; actually any $m$-symmetric scheme $Z$ contains $mP$ (this is quite obvious: if $f \in I_Z$, then every line thru $P$ meets $\{f = 0\}$ with multiplicity at least $m$, hence $f \in (I_P)^m$ ).

**Question:** What is the maximal lenght (if any) for an $m$-symmetric $Z$ ?

## 1. On some 0-dimensional schemes in the plane

**Example:** One simple example of an $m$-symmetric $Z$ which is not an $mP$ is the scheme given by the ideal $I_Z = (x^m, y^m)$, whose lenght is $\ell(Z) = m^2$. This of course shows that the maximal length is $\geq m^2$ .

**Definition:** An $m$-symmetric scheme of maximal length will be called an *m-superfat point*, or just a *superfat point* if we do not need to specify $m$.

**Definition:** An $m$-symmetric scheme whose ideal is of type $(L_1^m, L_2^m)$, for $L_i \in \mathbb{C}[x, y]_1$, $L_1 \neq L_2$, is called an *m-square point*, or just an *m-square*.

**Proposition:** Let $Z$ be an $m$-superfat point, then we have $\ell(Z) \leq m^2$; hence $\ell(Z) = m^2$. Moreover, for $m$=2, any 2-superfat point is a 2-square.

## 1. On some 0-dimensional schemes in the plane

**Sketch of proof:** Let $P$=(0,0) and $I_Z = (G_1,\dots,G_s)$; at least two of the $G_i$ must have multiplicity $m$ at $P$, let $(G_1,\dots,G_r)$ be the ones which have multiplicity $m$ at $P$, and $(F_1,\dots,F_r)$ their tangent cones, that we may assume to be linearly independent. Then a generic linear combination $F$ of $F_1,\dots, F_{r-1}$ will have no common factor with $F_r$, hence $(F, F_r)$ will define a 0-dimensional scheme $X$ at $P$ with $\ell(X) = m^2$ . Since $Z \subseteq X$, we have $\ell(Z) \leq \ell(X) = m^2$ . Thus, $\ell(Z) = m^2$ .

Case $m$=2. Since $\ell(Z) = 4$, there are two conics in the ideal, necessarily singular, say $L_1L_2$ and $L_3L_4$ . These two conics generate a line in the $\mathbb{P}^2$ parameterizing 2-forms in $x,y$; such line will intersect the conic which parameterizes squares of linear forms in two points, say $x^2$ and $y^2$, so we have $I_Z = (x^2,y^2)$.

## 1. On some 0-dimensional schemes in the plane

**Remarks:**

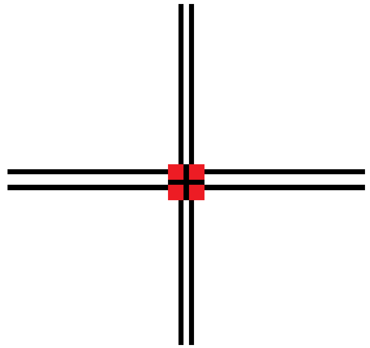
- There are other *m*-symmetric schemes either than fat points or superfat points, e.g the ideal $(x^3, y^3, x^2y^2)$ defines a 3-symmetric scheme $Z$ with $\ell(Z) = 8$.
- There are infinitely many different *m*-superfat points and *m*-squares with support at the same point (not true for fat points).
- Not all superfat points are squares, e.g. the ideal $(x^3+y^3, y^4- xy^2, xy^3, x^2y^2)$ defines a 3-superfat point, but it is not a c.i. It is generated as a generic scheme of length 9.

**Question:** Does a generic *m*-superfat point have generic Hilbert funcion? It looks likely, from examples. But what does "generic" mean here?

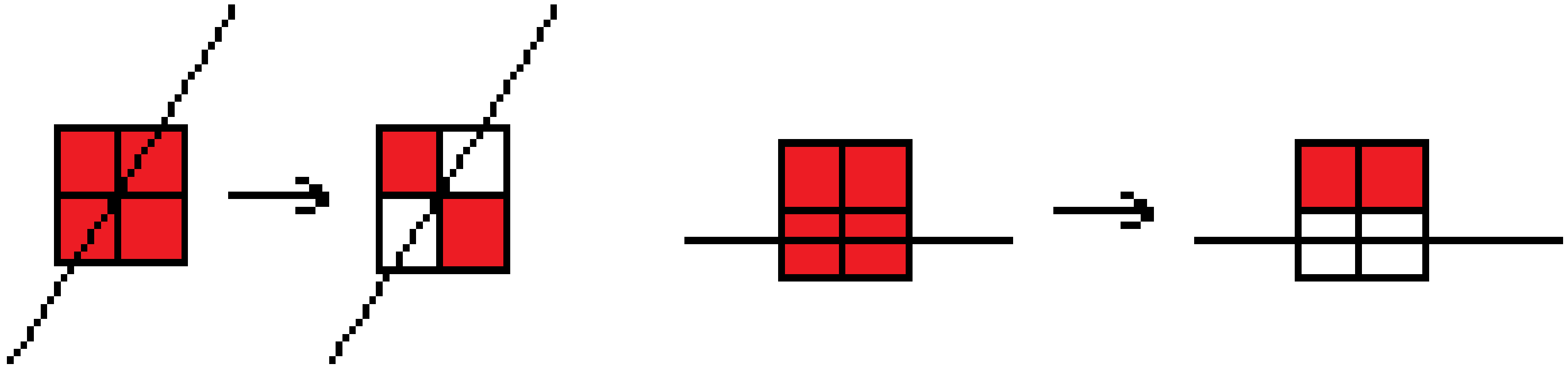
## 1. On some 0-dimensional schemes in the plane

Now, *superfat* points and in particular *m-squares*, have properties that again are counterintuitive: they are symmetric (can be viewed as «disks», as it happens for fat points) but, at the same time, they can have «particular directions»; let us consider the simplest ones, the 2-squares:

The 2-square of ideal $(x^2,y^2)$, has of course 2 particular directions, and



when we do the residual with respect to a generic line, we get a 2-jet on a different line; only for the lines $x=0$ and $y=0$ we have that the residual jet is on the same line.



## 1. On some 0-dimensional schemes in the plane

**Question:** What happens if we take all the *m*-square points with support at the same point?

**Proposition:** *For every* $P \in \mathbb{P}^2$, *we have that the schematic union of all m-squares supported at P is the fat point* $(2m\text{-}1)P$.

The proof is quite technical, involving mainly linear algebra and a combinatorial lemma.

**Remark:** The above is not true for particular unions; i.e. if you take all the 2-squares with ideal $(L_1^2, L_2^2)$, where $L_1$, $L_2$ are perpendicular lines thru the origin, then the union of all those gives a scheme of length 5, which has ideal $(x^2 + y^2, y^3, xy^2)$ . Notice that this union of 2-symmetric schemes is NOT 2-symmetric: $(x \mp iy)$ meets it with length 3!

## 1. On some 0-dimensional schemes in the plane

**Remark:** If we consider the scheme we just obtained, defined by the ideal $(x^2 + y^2, y^3, xy^2)$ , over R, then it is a 2-superfat point of length 5!

**Question:** What is the Hilbert function of generic unions of superfat points?

As we know this problem is open also for fat points, for this schemes we guess what the answer is for 2-squares:

**Conjecture:** The union $Z$ of $s$ generic 2-square points has the same generic Hilbert function as $4s$ simple points.

**Motivation:** Examples…But $s$=7 is tricky!

## 2. 2-squares and $2^n$-tensors

We want to explore what use superfat points can have (if any) on the study of structured tensors, namely on Symmetric and Partially Symmetric tensors of shape $2^n$, i.e. $\mathbb{C}^2 \times \mathbb{C}^2 \times \ldots \times \mathbb{C}^2$ ($n$-times).

In order to fix ideas, let's give a fast look of $2^4$-tensors of those kind.

General $2^4$-tensors $\longrightarrow$ points in $\mathbb{P}^{15}$

where tensors of rank 1 are parameterized by the Segre surface:

$V_{1,1,1,1}$, image of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ via the Segre map which is defined

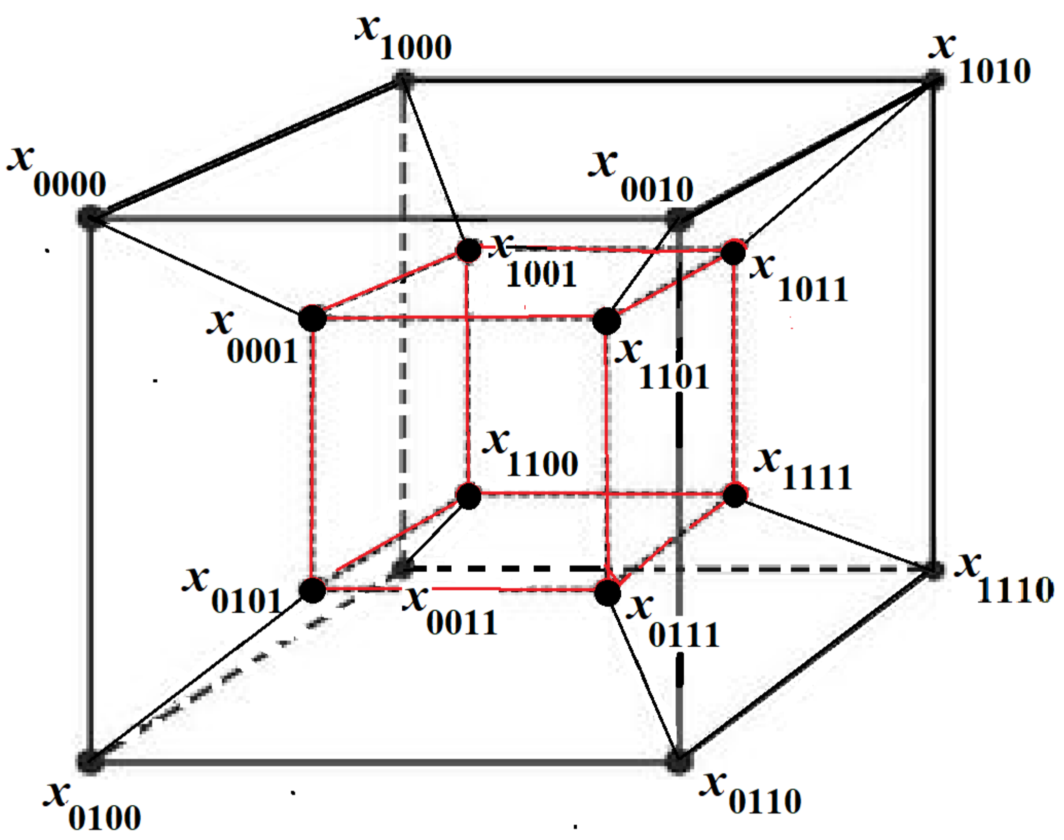
by the forms of multidegree (1,1,1,1) in $\mathbb{C}[s_0,s_1; t_0,t_1; u_0,u_1; v_0,v_1]$.

## 2. $2^n$-tensors

Here the $x_{ijkl}$ , $i,j,k,l \in \{0,1\}$ ,
are the cohordinates in $\mathbb{P}^{15}$ ;
the Segre Variety $V_{1,1,1,1} \subseteq \mathbb{P}^{15}$ ,
is parameterized by

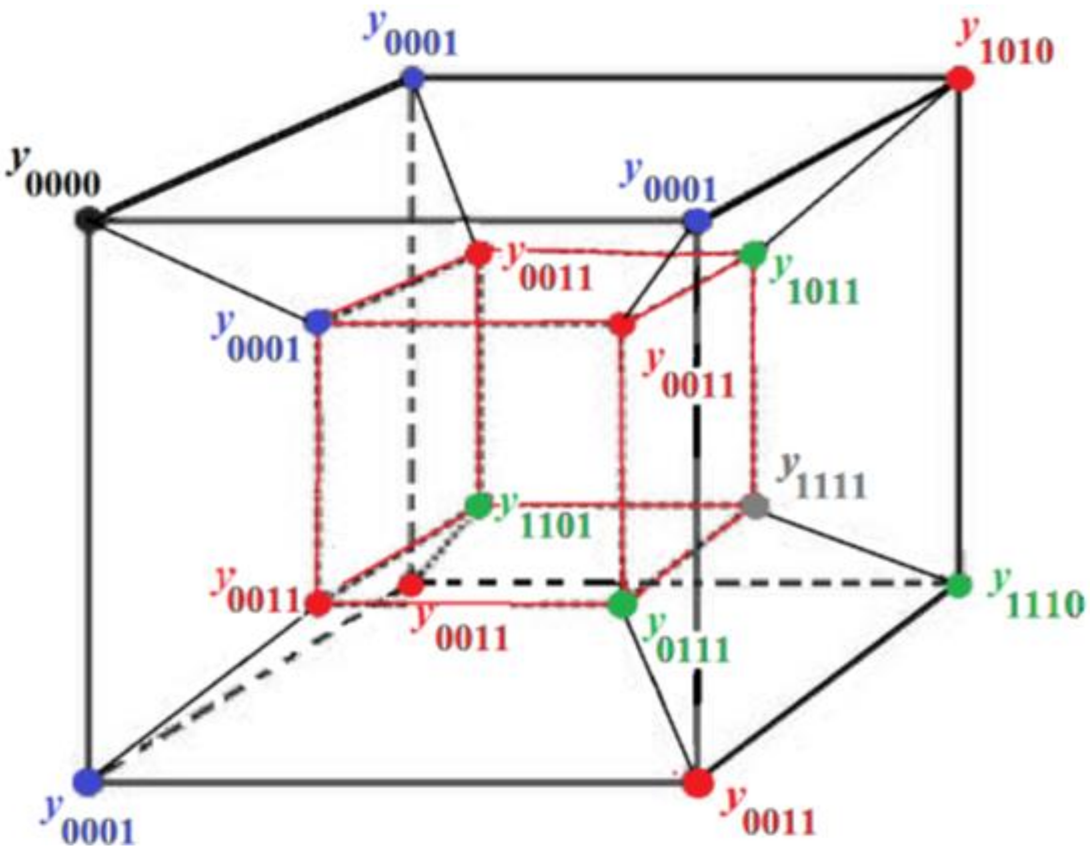
$x_{ijkl} = s_i t_j u_k v_l$ , $i,j,k,l \in \{0,1\}$ ,

and its ideal is defined by
the 2×2 minors in the tensor.



**2. $2^n$-tensors**

If we are interested in symmetric $2^4$-tensors (Bosons), we have to consider the space $\mathbb{P}^4 \subseteq \mathbb{P}^{15}$ defined by the symmetry relations: $x_{ijkl} = x_{\sigma(ijkl)}$, (for any permutation $\sigma$). We can use coordinates $y_{ijkl}$ in this $\mathbb{P}^4$ and here the tensors of *symrank* 1 are given by the Veronese embedding $\mathbb{P}^1 \rightarrow \mathbb{P}^4$ i.e. the rational normal curve $V_4$, defined again by the $2\times2$ minors in the tensor.



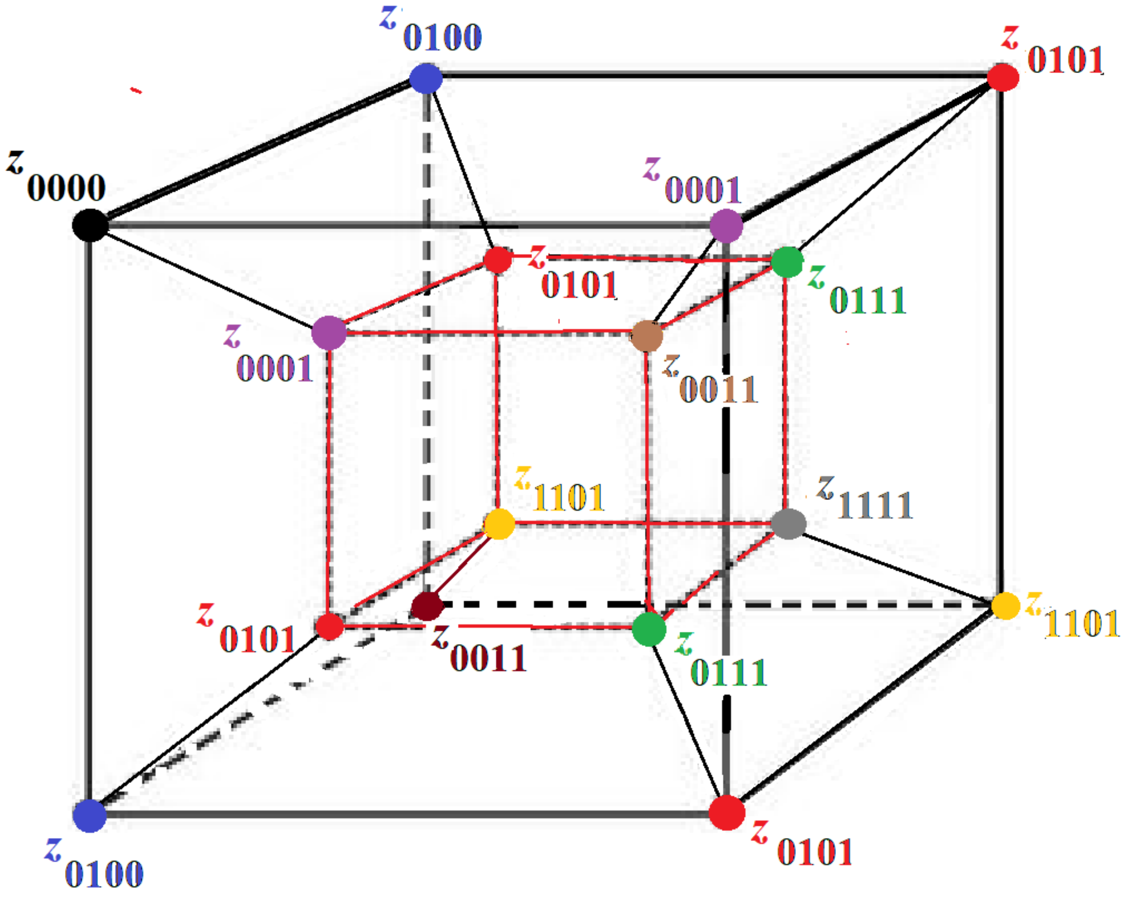

Symmetric $2^4$-tensors $\longleftrightarrow$ Forms of degree 4 in 2 variables.

## 2. $2^n$-tensors

There is an intermediate locus between the previous two; the space of partially symmetric tensors: we have:

$\mathbb{P}^4 \subseteq \mathbb{P}^8 \subseteq \mathbb{P}^{15}$ , where the $\mathbb{P}^8$ is given by the tensors $x_{ijkl}$ symmetric, separately, on the first and the second two indices: $x_{ijkl} = x_{\sigma(ij)\tau(kl)}$ (for all permutations $\sigma, \tau$ ), for which we have used coordinates $z_{ijkl}$

Here too the rank 1 tensors are given by all the $2\times 2$ minors in the tensor.



## 2. $2^n$-tensors

We can view this $\mathbb{P}^8$ space of partially symmetric $2^4$-tensors as the space $\mathbb{P}(\mathbb{C}[s_0,s_1;t_0,t_1]_{2,2}) = \mathbb{P}S^2(\mathbb{C}[s_0,s_1])\otimes S^2(\mathbb{C}[t_0,t_1])$, i.e. the (2,2)-forms, bihomogeneous of bidegree (2,2).

Under that identification, the forms of *ps*rank = 1 are parameterized by the Segre-Veronese variety $V_{2,2} \subseteq \mathbb{P}^8$, given by the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ via the (2,2)-forms, with parametric equations

$$z_{ijkl} \longrightarrow s_i s_j t_k t_l \in \mathbb{C}[s_0,s_1;\, t_0,t_1] \;,\; i,j,k,l \in \{0,1\}\,.$$

The surface $V_{2,2}$ is classically known as a particular Del Pezzo surface, the only one which is not a projection of the triple Veronese embedding of $\mathbb{P}^2$ from points on it.

## 2. 2-squares and partially symmetric tensors

Now back to 2-squares: if we consider a 2-square $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and then the Segre-Veronese embedding $\nu_{2,2}$ into $\mathbb{P}^8$, we have that $\nu_{2,2}(Z) \subseteq V_{2,2}$ will span a space $<\nu_{2,2}(Z)> \cong \mathbb{P}^3 \subseteq \mathbb{P}^8$, given by $(I_Z{}^{\perp})_{2,2}$.

If we want to consider the variety given by the union of all $<\nu_{2,2}(Z)>$ for $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we get:

$$\bigcup_{\{Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1\}} <\nu_{2,2}(Z)> = \bigcup_{\{P \in \mathbb{P}^1 \times \mathbb{P}^1\}} <\nu_{2,2}(3P)> = \tau_2(V_{2,2})$$

Where $\tau_2(V_{2,2})$ is the 2-osculating variety of $V_{2,2}$, i.e. the union of all the osculating $\mathbb{P}^5$'s to $V_{2,2}$.

In fact, the osculating $\mathbb{P}^5$ to a point $\nu_{2,2}(P)$ in $V_{2,2}$, is just $<\nu_{2,2}(3P)>$, and we have proved that the union of all $Z$ supported at $P$ is exactly $3P$.

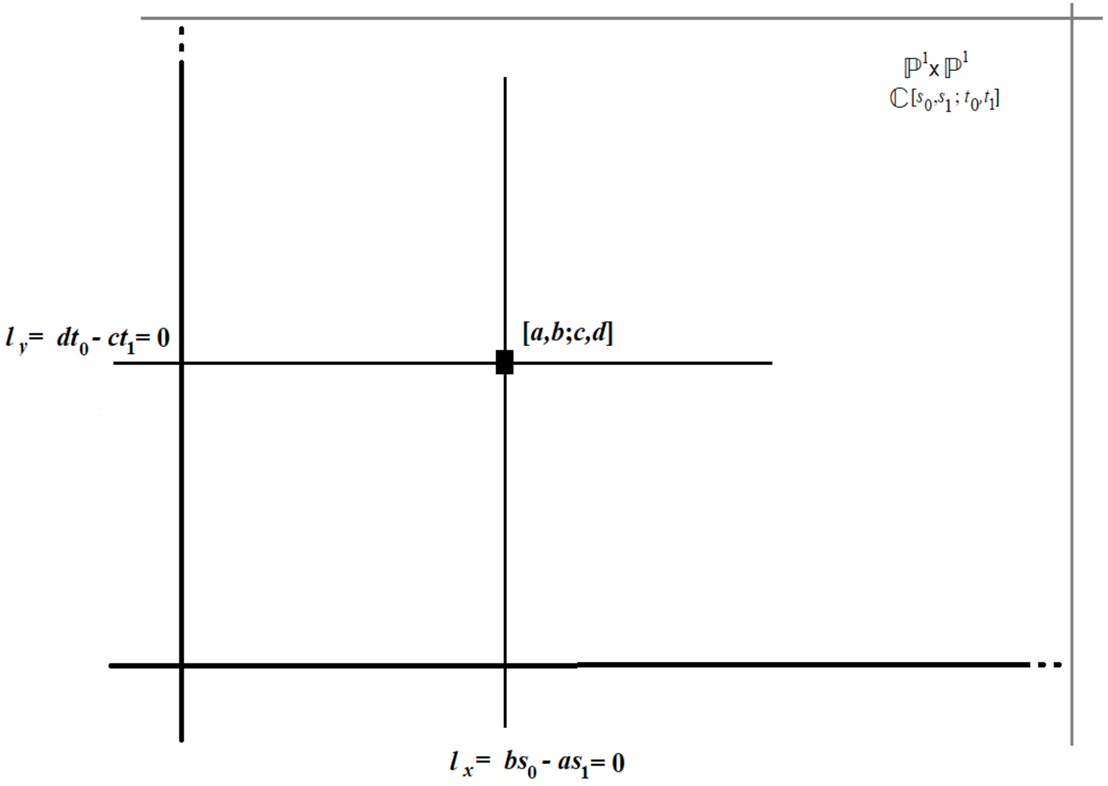
## 2. <u>2-squares and partially symmetric tensors</u>

So there is no reason to study the variety of all the $<v_{2,2}(Z)>$ 's , but if we consider a point in $\mathbb{P}^1 \times \mathbb{P}^1$ , there is a quite natural scheme $Z$ to be considered at $P = [a,b:c,d]$, namely let:

$l_s : \{bs_0\text{-}as_1=0\}$,   $l_t : \{dt_0\text{-}ct_1=0\}$ ,

$$I_Z = (l_s^2, l_t^2)$$

All other 2-squares which we may consider  (generated by other pairs of affine lines) give ideals generated in bidegree (2,2), while ours has two generator in bidegrees (0,2) and (2,0).



## 2. 2-squares and partially symmetric tensors

Hence, for each point $P = [a,b:c,d]$ in $\mathbb{P}^1 \times \mathbb{P}^1$, consider the scheme $Z(P)$, with $I_{Z(P)} = (l_s^2, l_t^2)$ and then the variety:

$$q_2(V_{2,2}) = \bigcup_{\{P \in \mathbb{P}^1 \times \mathbb{P}^1\}} \langle \nu_{2,2}(Z(P)) \rangle \quad .$$

We obviously have:

$$\tau_1(V_{2,2}) \subseteq q_2(V_{2,2}) \subseteq \tau_2(V_{2,2}) \text{ and } \dim q_2(V_{2,2}) = 5.$$

Since $\langle \nu_{2,2}(Z(P)) \rangle = (I_{Z(P)}{}^{\perp})_{2,2}$, let

$$m_s : \{as_0+bs_1=0\}, \quad m_t : \{ct_0+dt_1=0\} ,$$

so that $l_s \perp m_s$ and $l_t \perp m_t$ , with respect to the apolarity action.

**2. 2-squares and partially symmetric tensors**

We have $I_{Z(P)} = (l_s^2, l_t^2)_{2,2} = <l_s^2 l_t^2, l_s^2 m_t l_t, l_s^2 m_t^2, m_s^2 l_t^2, m_s l_s l_t^2>$ .

Then: $(I_Z^{\perp})_{2,2} = < m_s^2 m_t^2, m_s^2 m_t l_t, m_s l_s m_t^2, m_s l_s m_t l_t>$ .

---

We can recall that the tangent space to $V_{2,2}$ at $\nu_{2,2}(P)$ is

$$< m_s^2 m_t^2, m_s^2 m_t l_t, m_s l_s m_t^2> \ .$$

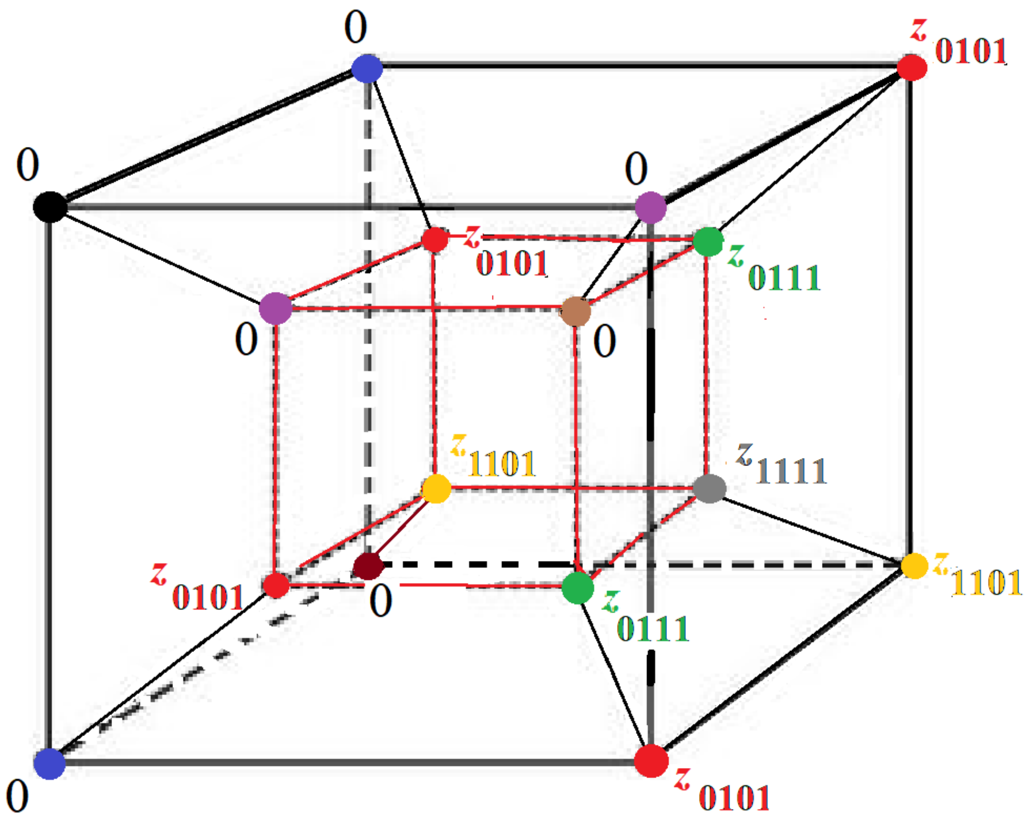
---

Hence, a point $T$ in $q_2(V_{2,2})$ can be written (by choosing appropriate coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ as:

$$z_{1111}\, s_1^2 t_1^2 + z_{1101}\, s_1^2 t_0 t_1 + z_{0111}\, s_0 s_1 t_1^2 + z_{0101}\, s_0 s_1 t_0 t_1$$

**2. <u>2-squares and partially symmetric tensors</u>**

So, $q_2(V_{2,2})$ parameterizes $2^4$-tensors which can be written as:



## 2. 2-squares and partially symmetric tensors

**Proposition:** $\sigma_2(q_2(V_{2,2})) = \mathbb{P}^8$, *as expected.*

**Proof**:

$<v_{2,2}(Z(P))> = < m_s^2 m_t^2 , m_s^2 m_t l_t , m_s l_s m_t^2, m_s l_s m_t l_t > = (m_s m_t)_{2,2}$

So, let $f : (\mathbb{C}[s_0,s_1]_1)^2 \times (\mathbb{C}[t_0,t_1]_1)^2 \longrightarrow q_2(V_{2,2})$, with:

$$f(m_s,n_s;m_t,n_t) = m_s n_s m_t n_t \in (m_s m_t)_{2,2} \subseteq q_2(V_{2,2}) .$$

To get the tangent space to $q_2(V_{2,2})$ in $m_s n_s m_t n_t$ we must compute:

$$\lim_{\lambda\to 0} \frac{d}{d\lambda} f[(m_s,n_s;m_t,n_t)+\lambda(u_s,v_s;u_t,v_t)] =$$

$$\lim_{\lambda\to 0} \frac{d}{d\lambda} [(m_s+\lambda u_s)(n_s+\lambda v_s)(m_t+\lambda u_t)(n_t+\lambda v_t)]$$

for a generic $(u_s,v_s;u_t,v_t)$.

**2. <u>2-squares and partially symmetric tensors</u>**

This gives:

$$W = < u_s n_s m_t n_t\ , m_s n_s u_t n_t\ , m_s v_s m_t n_t\ , m_s n_s m_t v_t >$$

Where $(u_s, v_s; u_t, v_t)$ can be any forms. This can be written

$$<m_s^2 m_t^2, m_s^2 m_t l_t\ , m_s l_s\, m_t^2, m_s l_s\, m_t l_t, m_s n_s l_t^2\ , l_s^2 m_t n_t > \ =$$

$$= (m_s m_t\ , m_s n_s\ , m_t n_t\ )_{2,2}$$

$W$ has (affine) dim = 6, as expected, and $W = (I_X)_{2,2}$, the bidegree 2,2 part of the ideal of $X$ which is made of three points (not on a fiber), namely:

$$m_s \cap m_t\ , m_s \cap n_s\ , m_t \cap n_t\ .$$

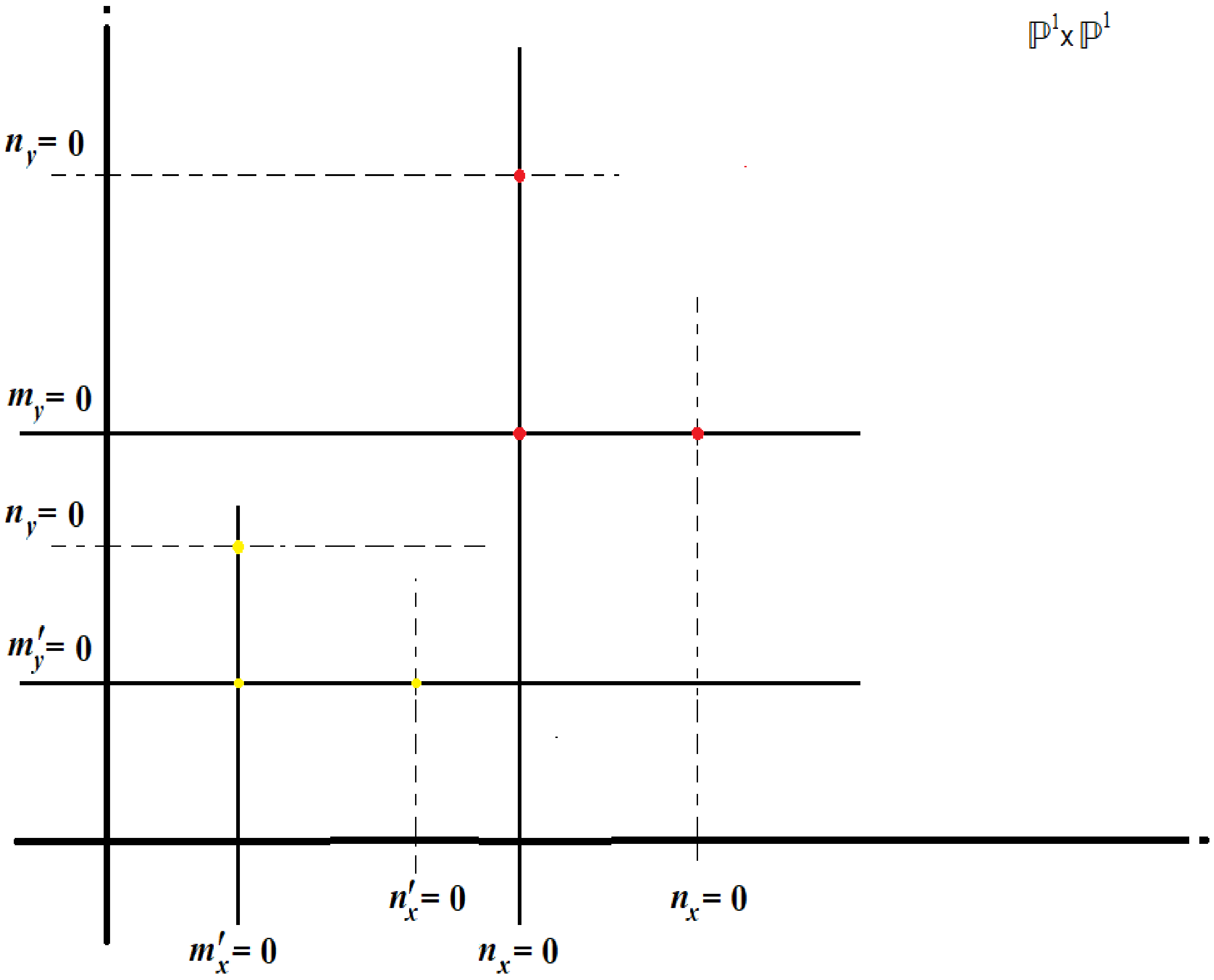
**2. <u>2-squares and partially symmetric tensors</u>**

By Terracini's Lemma, the affine cone on the tangent space to a generic point of $\sigma_2(q_2(V_{2,2}))$ is $W + W'$ where $W$ and $W'$ are the affine cones at the corresponding two points on $q_2(V_{2,2})$, hence :

$W + W' = (I_X)_{2,2} + (I_{X'})_{2,2} =$

$= (I_{X \cap X'})_{2,2} = R_{2,2}$ ,

Since $X \cap X' = \emptyset$.



## 2. 2-squares and partially symmetric tensors

The previous proposition can be generalized:

**Proposition:** $\dim \sigma_2(q_2(V_{d,d})) = 11$, *for all* $d \geq 3$, *as expected.*

**Proof**(sketch) :

Proceding as before, we get that

$<\nu_{d,d}(Z(P))> = < m_s^d m_t^d , m_s^d m_t^{d-1} l_t , m_s^{d-1} l_s m_t^d, m_s^{d-1} l_s m_t^{d-1} l_t> = (m_s^{d-1} m_t^{d-1})_{d,d}$

So, let $f : (\mathbb{C}[s_0,s_1]_1)^2 \times (\mathbb{C}[t_0,t_1]_1)^2 \longrightarrow q_2(V_{d,d})$, with:

$$f(m_s,n_s;m_t, n_t) = m_s^{d-1} n_s m_t^{d-1} n_t \in (m_s^{d-1} m_t^{d-1})_{d,d} \subseteq q_2(V_{d,d}) .$$

To get the tangent space to $q_2(V_{d,d})$ in $m_s^{d-1} n_s m_t^{d-1} n_t$ we must compute:

$$\lim_{\lambda \to 0} \frac{d}{d\lambda} f[(m_s,n_s;m_t, n_t)+\lambda(u_s,v_s;u_t, v_t)] =$$

$$\lim_{\lambda \to 0} \frac{d}{d\lambda} [(m_s+\lambda u_s)^{d-1}(n_s+\lambda v_s)(m_t+\lambda u_t)^{d-1}(n_t+\lambda v_t)]$$

for a generic $(u_s,v_s;u_t, v_t)$.

## 2. 2-squares and partially symmetric tensors

And we end up with

$$W = \langle m_s^d m_t^d,\ m_s^{d-1} l_s m_t^d,\ m_s^d m_t^{d-1} l_t,\ m_s^{d-1} l_s m_t^{d-1} l_t,\ m_s^{d-1} n_s m_t^{d-2} l_t^2,$$

$$m_s^{d-2} l_s^2 m_t^{d-1} n_t \rangle =$$

$$= (m_s^{d-1} m_t^{d-1},\ m_s^{d-1} n_s m_t^{d-2} l_t^2,\ m_s^{d-2} l_s^2 m_t^{d-1} n_t)_{d,d}$$

$W \subseteq (I_X)_{d,d}$, the bidegree $d,d$ part of the ideal

$$I_X = (m_s^{d-2} m_t^{d-2},\ m_s^{d-1} n_s,\ m_t^{d-1} n_t).$$

Where $W$ has (affine) dim =6, as expected, while dim $(I_X)_{d,d} = 7$, and $X$ is a scheme which is made of the lines $m_s$ and $m_t$ ($d$-2)ple and two (d-1)-jets on the lines $n_s$, $n_t$ at the points $m_s \cap n_s$, $m_t \cap n_t$.

**2. <u>2-squares and partially symmetric tensors</u>**

As in the case $d$=2, dim $\sigma_2(q_2(V_{d,d}))$ will be given by $W + W'$, where $W$ and $W'$ are the affine cones on two generic points on $q_2(V_{d,d})$, and :

$$\dim(W + W') = \dim W + \dim W' - \dim W \cap W'.$$

We have $W \cap W' \subseteq (I_X)_{d,d} \cap (I_{X'})_{d,d} = (I_{X\cup X'})_{d,d} = \{0\}$ , since the forms in $(I_{X\cup X'})_{d,d}$ should contain $m_s^{d-2} m_t^{d-2} m'^{d-2}_s m'^{d-2}_t$ and have intersection $d$-1 with the lines $n_s$ ,$n_t$ ,$n'_s$ ,$n'_t$ at the four points $m_s \cap n_s$ , $m_t \cap n_t$ , $m'_s \cap n'_s$ , $m'_t \cap n'_t$. This is impossible for every $d \geq 3$ . Hence:

$$\dim(W + W') = \dim W + \dim W' = 6 + 6 = 12,$$

And so dim $\sigma_2(q_2(V_{d,d})) = 11$.

### 3. $q_2(V_{d,d})$ , *W*-states and cuckoo varieties

There is an interesting subvariety of $q_2(V_{d,d})$, expecially for application to Quantum Physics. Every Segre-Veronese variety

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1 \xrightarrow{d,d,\dots,d} \mathbb{P}^{d^k-1}$$

Is related to the study of *entanglement* of $k$ $d$-body systems made of different species of indistinguishable bosonic particles (like photons): the Hilbert space parameterizing their states is

$$S^d\mathcal{H}_1 \otimes S^d\mathcal{H}_2 \dots \otimes S^d\mathcal{H}_k ,$$

and in the case $\mathcal{H}_k = \mathbb{C}^2$, it is of interest to look at tensors (*W*-states in physics) of type $W_d \otimes \dots \otimes W_d$, where

$$W_d = x^{d-1}y \in S^d\,\mathbb{C}^2 \subseteq (\mathbb{C}^2)^{\otimes d}$$

(e.g. see Ballico, Bernardi, Christandl, Gesmundo 2019)

## 3. $q_2(V_{d,d})$ , *W*-states and cuckoo varieties

Let us consider the case $k$=2;

$q_2\,(V_{d,d})$ parameterizes partially symmetric tensors given by :

$<\nu_{d,d}(Z(P))> \;=\; < m_s^d m_t^d\,,\, m_s^d m_t^{d-1}\, l_t\,,\, m_s^{d-1} l_s\, m_t^d,\, m_s^{d-1} l_s m_t^{d-1} l_t> \;=$

$$= (m_s^{d-1} m_t^{d-1})_{\,d,d}$$

Hence for all tensors of type

$$W_d \otimes W_d = m_s^{d-1} a_s m_t^{d-1} b_t \text{ , for } a_s \in R_{1,0} \text{ , } b_t \in R_{0,1}$$

We have: $\quad W_d \otimes W_d \in q_2\,(V_{d,d})$.

More specifically, consider the subvariety $qq_2\,(V_{d,d}) \subseteq q_2\,(V_{d,d})$ which parameterizes exactly the tensors $W_d \otimes W_d$ (this is the cuckoo variety).

## 3. $q_2(V_{d,d})$ , *W*-states and cuckoo varieties

$qq_2(V_{d,d})$ is the image of the map:

$\mathbb{P}^{1*}\times\mathbb{P}^{1*}\times\mathbb{P}^{1*}\times\mathbb{P}^{1*} \longrightarrow q_2(V_{d,d})$ , with: $(m_s, a_s, m_t, b_t) \longrightarrow m_s^{d-1}a_s m_t^{d-1}b_t$

(For given $m_s$, $m_t$ the image is a quadric inside $<v_{d,d}(Z(P))> \cong \mathbb{P}^3$ ).

So $qq_2(V_{d,d})$ has dimension 4 and (for $d>2$) it is isomorphic to $Q\times Q$ , where $Q$ is a smooth quadric in $\mathbb{P}^3$.

In the case $d=2$, it is not hard to compute the ideal of $qq_2(V_{2,2})$ into $\mathbb{P}^8$; it is given by the $2\times2$- minors of the matrix:

$$M_2 = \begin{pmatrix} z_{0000} & z_{0001} & z_{0011} \\ z_{0100} & z_{0101} & z_{0111} \\ z_{1100} & z_{1101} & z_{1111} \end{pmatrix}.$$

Which shows that $qq_2(V_{2,2})$ is just the Segre variety $s_{1,1}(\mathbb{P}^2\times\mathbb{P}^2)$; actually it is easy to check that $qq_2(V_{2,2}) = s_{1,1}(\tau(V_2) \times \tau(V_2)) = s_{1,1}(\mathbb{P}^2\times\mathbb{P}^2)$.

## 3. $q_2(V_{d,d})$ , $W$-states and cuckoo varieties

Anyway, this can be generalized:

**Remark:** *Any* $qq_2(V_{d,d}) = s_{1,1}(\tau(V_d) \times \tau(V_d)) \subseteq s_{1,1}(\mathbb{P}^d \times \mathbb{P}^d)$ .

And $qq_2(V_{d,d})$ is 2-defective, but not as much as $s_{1,1}(\mathbb{P}^d \times \mathbb{P}^d)$ :

**Proposition:** *Any* $\sigma_2(qq_2(V_{d,d}))$ , $d>1$ *is defective, with defect* $=2$ .

This can be seen via the standard way of Terracini's Lemma, studying the tangent space in two generic points of the variety. Moreover, trivially:

**Proposition:** The ideal of $\sigma_2(qq_2(V_{d,d}))$ is given by the $2\times 2$- minors of the $(d+1)\times(d+1)$-minors of the matrix $M_d$ which has the $(d+1)^2$ variables of the space $S^d\,\mathbb{C}^2$ as entries and by the relations among the entries on each row which define $\tau(V_d)$ inside $\mathbb{P}^d$.

The generalization to the case of more factors is on its way …
The proof that $m$-superfat points have degree $m^n$ works also in $\mathbb{A}^n$.