Singular points from subsecant loci of k-secants of Veronese varieties

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This talk is based on an on-going project with:

- K. Furukawa (Josai Univ.) (arXiv:2111.03254).
- (a new version will be appeared soon)

Secant variety

In the talk, we work over $\mathbb{C}$, the field of complex numbers.

Let $X \subset \mathbb{P}^N$ be an embedded projective variety. The k-th *secant variety* of X is defined as:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\dots,x_k \in X} \langle\, x_1, \dots, x_k \,\rangle} \subset \mathbb{P}^N \ ,$$

where $\langle\, x_1, \dots, x_k \,\rangle \subset \mathbb{P}^N$ denotes the linear span of the points $x_1, \dots, x_k$ and the overline means the Zariski closure.

In particular, $\sigma_1(X) = X$ and $\sigma_2(X)$ has been often called just *secant* or *secant line* variety of X in the literature.

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This secant variety construction (more generally, the *join* construction of subvarieties) is one of the most famous methods in classical algebraic geometry. You can make a new variety from an old one.

But, still many fundamental questions for k-secants are open (e.g. dimension, degree, singularity, equation and syzygies, identifiability, etc).

Singular loci of secant variety

Today, we consider singular loci of $\sigma_k(v_d\mathbb{P}V)$, $\mathrm{Sing}(\sigma_k(v_d\mathbb{P}V))$.

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Today, we consider singular loci of $\sigma_k(v_d\mathbb{P}V)$, $\mathrm{Sing}(\sigma_k(v_d\mathbb{P}V))$.

Knowledge on singularities of higher secant varieties is worthwhile and important itself in the study of algebraic geometry.

Further, it is also known to be closely related to the *identifiability* problem of structured tensors in applications.

Singular loci of secant variety

<u>Terracini's lemma</u> For an irreducible variety $X \subset \mathbb{P}V$ and for any $x_1, \ldots, x_k \in X$ and any $y \in \langle x_1, \cdots, x_k \rangle$, we have

$$\mathbb{T}_y \sigma_k(X) \supset \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_k} X \rangle$$

and "$=$" holds for general choices of $x_1, \ldots, x_k$ and y.

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Proposition $\mathrm{Sing}(\sigma_{k+1}(X)) \supset \sigma_k(X)$ unless $\sigma_{k+1}(X)$ is linear.

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Proposition $\quad \mathrm{Sing}(\sigma_{k+1}(X)) \supset \sigma_k(X)$ unless $\sigma_{k+1}(X)$ is linear.

Proof) Choose any $y \in \sigma_k(X)$. Then, by Terracini's lemma, we see that for any $x \in X$

$$\begin{aligned} &\mathbb{T}_y \sigma_{k+1}(X) \supset \langle \mathbb{T}_y \sigma_k(X), \mathbb{T}_x X \rangle \\ &\Rightarrow \mathbb{T}_y \sigma_{k+1}(X) \supset \langle y, X \rangle \supset \langle X \rangle = \langle \sigma_{k+1}(X) \rangle \end{aligned}$$

Since $\sigma_{k+1}(X)$ is not linear, $\dim \mathbb{T}_y \sigma_{k+1}(X) > \dim \sigma_{k+1}(X)$ for any $y \in \sigma_k(X)$.

Known results

As mentioned above, we have

$$\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}^n))) \supseteq \sigma_{k-1}(v_d(\mathbb{P}^n)) \ .$$

We call a point $p \in \sigma_k(X)$ non-trivial singular point if $p \notin \sigma_{k-1}(X)$ and $\sigma_k(X)$ is singular at p, while the points belonging to $\sigma_{k-1}(X)$ are called trivial singularity.

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- First, it is classical that "$=$" is true for the binary case (i.e. $n = 1$)
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- First, it is classical that "$=$" is true for the binary case (i.e. $n = 1$)
- Also true for symmetric matrices (i.e. he case of quadratic forms (i.e. $d = 2$)
- ($k = 2$) Kanev proved that this equality holds for any d, n.
- ($k = 3$, H. 2018) It was shown in that a non-trivial singularity can happen and it is only when $d = 4$ and $n \geq 3$.

Singular loci of k-secant of $v_d(\mathbb{P}^n)$

To sum up, we have

$(\mathbf{k}, \mathbf{d}, \mathbf{n})$	**Singular locus of** $\sigma_k(v_d(\mathbb{P}^n))$
$(\geq 2, \geq 2, 1)$	σ_{k-1}
$(\geq 2, 2, \geq 1)$	σ_{k-1}
$(2, \geq 2, \geq 1)$	σ_1
$(3, 3, \geq 2)$	σ_2
$(3, \geq 4, 2)$	σ_2
$(3, 4, \geq 3)$	$\mathcal{D} \cup \sigma_2$
$(3, \geq 5, \geq 3)$	σ_2
$(\geq 4, \geq 3, \geq 2)$	??? Try to answer this!

Figure: Singular locus of $\sigma_k(v_d(\mathbb{P}^n))$

Remark on equations for $\sigma_k(v_d(\mathbb{P}^n))$

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In principle, once a complete set of equations for higher secant variety is attained, it is straightforward to check the singularity in some sense (despite of its computational complexity).

But, only a few has been known on the defining equations of higher secant varieties and it seems far from being understood completely at this moment, even for the Veronese case (see the Table due to Landsberg-Ottaviani for the state of art).

Remark on equations for $\sigma_k(v_d(\mathbb{P}^n))$

Case	Equations	Cuts out	References
$\sigma_r(v_2(\mathbb{P}^n))$	Size $r+1$ minors	Ideal	Classical
$\sigma_r(v_d(\mathbb{P}^1))$	Size $r+1$ minors of any $\phi_{s,d-s}$	Ideal	Gundelfinger, [25]
$\sigma_2(v_d(\mathbb{P}^n))$	Size 3 minors of any $\phi_{1,d-1}$ and $\phi_{2,d-2}$	Ideal	[26]
$\sigma_3(v_3(\mathbb{P}^n))$	Aronhold + size 4 minors of $\phi_{1,2}$	Ideal	Proposition 2.3.1 Aronhold for n=2[25]
$\sigma_3(v_d(\mathbb{P}^n)), d \geq 4$	Size 4 minors of $\phi_{2,2}$ and $\phi_{1,3}$	Scheme	Theorem 3.2.1 (1) [46] for $n=2, d=4$
$\sigma_4(v_d(\mathbb{P}^2))$	Size 5 minors of $\phi_{a,d-a}, a=\lfloor\frac{d}{2}\rfloor$	Scheme	Theorem 3.2.1 (2) [46] for $d=4$
$\sigma_5(v_d(\mathbb{P}^2)), d \geq 6$ and $d=4$	Size 6 minors of $\phi_{a,d-a}, a=\lfloor\frac{d}{2}\rfloor$	Scheme	Theorem 3.2.1 (3) Clebsch for $d=4$[25]
$\sigma_r(v_5(\mathbb{P}^2)), r \leq 5$	Size $2r+2$ subPfaffians of $\phi_{31,31}$	Irred. comp.	Theorem 4.2.7
$\sigma_6(v_5(\mathbb{P}^2))$	Size 14 subPfaffians of $\phi_{31,31}$	Scheme	Theorem 4.2.7
$\sigma_6(v_d(\mathbb{P}^2)), d \geq 6$	Size 7 minors of $\phi_{a,d-a}$, $a=\lfloor\frac{d}{2}\rfloor$	Scheme	Theorem 3.2.1 (4)
$\sigma_7(v_6(\mathbb{P}^2))$	Symm.flat. + Young flat.	Irred.comp.	Theorem 4.2.9
$\sigma_8(v_6(\mathbb{P}^2))$	Symm.flat. + Young flat.	Irred.comp.	Theorem 4.2.9
$\sigma_9(v_6(\mathbb{P}^2))$	det $\phi_{3,3}$	Ideal	Classical
$\sigma_j(v_7(\mathbb{P}^2)), j \leq 10$	Size $2j+2$ subPfaffians of $\phi_{41,41}$	Irred.comp.	Theorem 1.2.3
$\sigma_j(v_{2\delta}(\mathbb{P}^2)), j \leq \binom{\delta+1}{2}$	Rank$\phi_{a,d-a} = \min(j, \binom{a+2}{2})$, $1 \leq a \leq \delta$ open and closed conditions	Scheme	[25, Theorem 4.1A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^2)), j \leq \binom{\delta+1}{2}+1$	Rank$\phi_{a,d-a} = \min(j, \binom{a+2}{2})$, $1 \leq a \leq \delta$ open and closed conditions	Scheme	[25, Theorem 4.5A]
$\sigma_j(v_{2\delta}(\mathbb{P}^n)), j \leq \binom{\delta+n-1}{n}$	Size $j+1$ minors of $\phi_{\delta,\delta}$	Irred. comp.	[25, Theorem 4.10A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^n)), j \leq \binom{\delta+n}{n}$	Size $\binom{n}{a}j+1$ minors of $Y_{d,n}$, $a=\lfloor n/2\rfloor$ if $n=2a$, a odd, $\binom{n}{a}j+2$ subpfaff. of $Y_{d,n}$	Irred. comp.	Theorem 1.2.3

Remark on equations for $\sigma_k(v_d(\mathbb{P}^n))$

Thus, with limited knowledge on the equations for secant varieties, it is very difficult to determine singular loci in general.

So far, there has been no general idea or clear picture on the singular loci of higher secant varieties of Veronese varieties yet.

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So far, there has been no general idea or clear picture on the singular loci of higher secant varieties of Veronese varieties yet.

In this work, we introduce some way to contribute this as investigating

" Singular points from subsecant loci "

It is mainly based on some geometric ideas on embedded tangent spaces (Geometry) and on conormal space computations via flattenings (Algebra).

It helps us to make a more visible picture on the singular loci of higher secants.

Back to 3-secant case

Theorem (H. 2018)

Let $X = v_d(\mathbb{P}^n) \subset \mathbb{P}^N, N = \binom{n+d}{n} - 1, d \geq 3$. $\mathrm{Sing}(\sigma_3(X)) = \sigma_2(X)$ *except* $d = 4, n > 2$. *In the exceptional case,* $\mathrm{Sing}(\sigma_3(X)) = \sigma_2(X) \cup \mathcal{D}$, *where* $\mathcal{D} \not\subset \sigma_2(X)$ *is the locus of all degenerate forms in* $\sigma_3(X)$.

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For a $\mathbb{P}^1 \subset \mathbb{P}^n$, we have

$$v_d(\mathbb{P}^1) \subset \sigma_3(v_d(\mathbb{P}^1)) \subset \mathbb{P}^d = \langle v_d(\mathbb{P}^1) \rangle \subset \mathbb{P}^N \ .$$

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- $d > 4$: $\mathbb{P}^d \supsetneq \sigma_3(v_d(\mathbb{P}^1))$
- $d = 4$: $\mathbb{P}^4 = \sigma_3(v_4(\mathbb{P}^1)) \supsetneq \sigma_2(v_4(\mathbb{P}^1))$
- $d < 4$: $\mathbb{P}^d = \sigma_3(v_d(\mathbb{P}^1)) = \sigma_2(v_d(\mathbb{P}^1)) \subset \sigma_2(X)$

$\Rightarrow$ Trichotomy happens!

Geometric description for the singular loci

From the previous theorem, we can give a geometric description for the singular loci of 3-th secant of $\nu_4(\mathbb{P}V)$ in the exceptional case (i.e. $\dim V \geq 4$) as cone locus.

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$$\begin{aligned}
&\mathrm{Sing}(\sigma_3(\nu_4(\mathbb{P}V))) \\
&= \sigma_2(\nu_4(\mathbb{P}V)) \cup \{ \bigcup_{\mathbb{P}^1 \subset \mathbb{P}V} \sigma_3(\nu_4(\mathbb{P}^1))\} = \bigcup_{\mathbb{P}^1 \subset \mathbb{P}V} \langle \nu_4(\mathbb{P}^1) \rangle \\
&= \{f \in \mathbb{P}S^4V \mid f \text{ is any form which can be expressed using at most 2 vars}\}
\end{aligned}$$

which is an irreducible $2\dim V$-dimensional locus in the $(3\dim V - 1)$-dimensional variety $\sigma_3(\nu_4(\mathbb{P}V))$.

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This kind of a simple description of the singular locus can be attained in few more cases.

$\Rightarrow$ e.g. $\mathrm{Sing}(\sigma_4(\nu_3(\mathbb{P}^4)))$ is *the locus of cubic forms with the number of essential variables* ≤ 3.

Some terminology

For any given point $p \in \sigma_k(\nu_d(\mathbb{P}^n))$, as considering $(1, d-1)$-symmetric flattening, it is easy to see that for some $1 \leq m \leq k-1$, $\exists$ an m-plane $\mathbb{P}^m \subset \mathbb{P}^n$ such that $p \in \sigma_k(\nu_d(\mathbb{P}^m))$.

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So, we call $\sigma_k(\nu_d(\mathbb{P}^m))$ an m-subsecant variety of $\sigma_k(\nu_d(\mathbb{P}^n))$ if $m < k-1$ & $m < n$ i.e. $m < \min\{k-1, n\}$ (often call it simply a *subsecant variety*).

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We also call the union of all subsecant varieties the subsecant loci of the given k-th secant $\sigma_k(\nu_d(\mathbb{P}^n))$, whereas $p \in \sigma_k(\nu_d(\mathbb{P}^m))$ is called a point of full-secant loci if $m = \min\{k-1, n\}$ and p does not belong both to the subsecant loci and to the previous $\sigma_{k-1}(\nu_d(\mathbb{P}^n))$.

$\Rightarrow$ For $k = 3$, the only exceptional case on the singular loci of $\sigma_3(\nu_d(\mathbb{P}^n))$ is given as *the subsecant loci* in it.

Main result: $m = 1$ case

For 1-subsecant loci (i.e. $m = 1$), we obtain

Theorem A (Furukawa and H.)

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple Veronese embedding with $d \geq 3$ and $N = \binom{n+d}{d} - 1$. If $n \geq 2, k \geq 4$ or $n \geq 3, k \geq 3$, then it holds that

(i) *$\sigma_k(\nu_d(\mathbb{P}^n))$ is smooth at every point in $\sigma_k(\nu_d(\mathbb{P}^1)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ if $k \leq \frac{d+1}{2}$,*

(ii) *$\sigma_k(\nu_d(\mathbb{P}^1)) \subset \mathrm{Sing}(\sigma_k(\nu_d(\mathbb{P}^n)))$, but $\sigma_k(\nu_d(\mathbb{P}^1)) \not\subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ (i.e. non-trivial singularity) if $k = \frac{d+2}{2}$ (only happens if d is even),*

(iii) *$\sigma_k(\nu_d(\mathbb{P}^1)) \subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ (i.e. trivial singularity) if $k \geq \frac{d+3}{2}$,*

else if $n = 2, k = 3$, then it holds that

(iv) *$\sigma_3(\nu_d(\mathbb{P}^2))$ is smooth at every point in $\sigma_3(\nu_d(\mathbb{P}^1)) \setminus \sigma_2(\nu_d(\mathbb{P}^2))$ when $d \geq 4$ and $\sigma_3(\nu_d(\mathbb{P}^1)) \subset \sigma_2(\nu_d(\mathbb{P}^2))$ in case $d \leq 3$.*

$\Rightarrow$ This is a generalization of the result for $k = 3$ into any higher secant of Veronese varieties.
$\Rightarrow$ It also shows an interesting '*trichotomy*' phenomenon.
$\Rightarrow$ (iv) is the exceptional case to the trichotomy (singular $\to$ smooth when $d = 4$)

Main result: $m = 2$ case

Theorem A' (Furukawa and H.)

Let $d \geq 3, n \geq 3$, $N = \binom{n+d}{d} - 1$ and $k \geq 4$. If $(k, n) \neq (4, 3)$, then

(i) *$\sigma_k(\nu_d(\mathbb{P}^n))$ is smooth at a general point in $\sigma_k(\nu_d(\mathbb{P}^2)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ if $k \leq \frac{\binom{d+2}{2}}{3}$ with $d \neq 4, 6$, $k = 4$ when $d = 4$, or $k \leq 8$ when $d = 6$,*

(ii) *$\sigma_k(\nu_d(\mathbb{P}^2)) \subset \mathrm{Sing}(\sigma_k(\nu_d(\mathbb{P}^n)))$, but $\sigma_k(\nu_d(\mathbb{P}^2)) \not\subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ (i.e. non-trivial singularity) if $\frac{\binom{d+2}{2}}{3} < k < \frac{\binom{d+2}{2}}{3} + 1$ with $d \neq 4, 6$, $5 \leq k \leq 6$ when $d = 4$, or $k = 9$ when $d = 6$,*

(iii) *$\sigma_k(\nu_d(\mathbb{P}^2)) \subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ if $k \geq \frac{\binom{d+2}{2}}{3} + 1$ with $d \neq 4, 6$, $k \geq 7$ when $d = 4$, or $k \geq 10$ when $d = 6$,*

else if $k = 4, n = 3$, then it holds that

(iv) *$\sigma_4(\nu_d(\mathbb{P}^3))$ is smooth at every point in $\sigma_4(\nu_d(\mathbb{P}^2)) \setminus \sigma_3(\nu_d(\mathbb{P}^3))$ when $d = 3$ and $\sigma_4(\nu_d(\mathbb{P}^3))$ is smooth at a general point in $\sigma_4(\nu_d(\mathbb{P}^2)) \setminus \sigma_3(\nu_d(\mathbb{P}^3))$ in case $d \geq 4$.*

$\Rightarrow$ The 'general point condition' in (i) can not be removed.
$\Rightarrow$ $d = 3$ in (iv) also shows exceptional behavior to the trichotomy (checked via equations from prolongation!).

Main result : non-defective subsecant

For an m-subsecant $\sigma_k(\nu_d(\mathbb{P}^m))$ of $\sigma_k(\nu_d(\mathbb{P}^n))$ with any $m \geq 2$, we prove

Theorem B

Let $X = \sigma_k(\nu_d(\mathbb{P}^n))$, $n \geq 3$, $d \geq 3$, $k \geq 4$, and let $\mathbb{P}^m \subset \mathbb{P}^n$ be an m-plane with $2 \leq m < \min\{k-1, n\}$. Assume that (d, m) is not one of $(3,4), (4,2), (4,3)$ or $(4,4)$. Then, it holds that

(i) *X is smooth at a general point in $\sigma_k(\nu_d(\mathbb{P}^m)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ if $k \leq \frac{\binom{m+d}{m}}{m+1}$ unless $(k, d, m) = (9,6,2), (9,3,5)$.*

(ii) *$\sigma_k(\nu_d(\mathbb{P}^m))$ is a non-trivial singular locus of $\sigma_k(\nu_d(\mathbb{P}^n))$ (i.e. $\sigma_k(\nu_d(\mathbb{P}^m)) \subset \mathrm{Sing}(\sigma_k(\nu_d(\mathbb{P}^n)))$, but $\sigma_k(\nu_d(\mathbb{P}^m)) \not\subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$) if $\frac{\binom{m+d}{m}}{m+1} < k < \frac{\binom{m+d}{m}}{m+1} + 1$ or $(k, d, m) = (9,6,2), (9,3,5)$ <u>and if a condition $(*)$ is satisfied</u>. In particular, this does not appear if the divisibility $\frac{\binom{m+d}{m}}{m+1} \in \mathbb{N}$ is satisfied.*

(iii) *$\sigma_k(\nu_d(\mathbb{P}^m)) \subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ (i.e. trivial singularity of $\sigma_k(\nu_d(\mathbb{P}^n))$) if $k \geq \frac{\binom{m+d}{m}}{m+1} + 1$ or $(k, d, m) = (10,6,2), (10,3,5)$.*

Main result : non-defective subsecant

The condition $(*)$ is

$$(1)\ n \geq m+2 \quad \text{or} \quad (2)\ n = m+1 \ \&\ \binom{m+d-2}{m} \geq k-m+2\,.$$

- If $m=2, d\geq 4$ or $m\geq 3, d\geq \frac{3+\sqrt{4m+5}}{2}$, then $\binom{m+d-2}{m}\geq k-m+2$ is satisfied under the situation of (ii).
- Further, (d,m) also should satisfy $m\leq k-2$ and the condition of (ii), $\frac{\binom{m+d}{m}}{m+1} < k < \frac{\binom{m+d}{m}}{m+1}+1$. So, the condition $(*)$ is not valid only for very few (d,m)'s.
- For instance, if $m\leq 4$, there is only one case $(d,m)=(3,2)$ out of $(*)$, which gives another exceptional case $(k,d,n,m)=(4,3,3,2)$ to the trichtomy.

Main result : defective subsecant

For the cases of $(d,m) = (3,4),(4,2),(4,3)$ or $(4,4)$, we have the following theorem.

Theorem C

In the same situation as Theorem 1, if $(d,m) = (3,4),(4,2),(4,3)$ *or* $(4,4)$*, then we have*

(i) $\sigma_k(\nu_d(\mathbb{P}^n))$ *is smooth at a general point in* $\sigma_k(\nu_d(\mathbb{P}^m)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ *if* $k \leq \lfloor \frac{\binom{m+d}{m}}{m+1} \rfloor - 1$.

(ii) $\sigma_k(\nu_d(\mathbb{P}^m))$ *is a non-trivial singular locus of* $\sigma_k(\nu_d(\mathbb{P}^n))$ *if* $\lfloor \frac{\binom{m+d}{m}}{m+1} \rfloor \leq k \leq \lceil \frac{\binom{m+d}{m}}{m+1} \rceil + 1$.

(iii) $\sigma_k(\nu_d(\mathbb{P}^m)) \subset \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ *(i.e. trivial singularity) if* $k \geq \lceil \frac{\binom{m+d}{m}}{m+1} \rceil + 2$.

Main result : defective subsecant

Theorem C

$(\mathbf{d}, \mathbf{m})$	*Generic smooth* (i)	*Non-trivial Sing.* (ii)	*Trivial Sing.* (iii)
$(3, 4)$	$k \leq 6$	$7 \leq k \leq 8$	$k \geq 9$
$(4, 2)$	$k \leq 4$	$5 \leq k \leq 6$	$k \geq 7$
$(4, 3)$	$k \leq 7$	$8 \leq k \leq 10$	$k \geq 11$
$(4, 4)$	$k \leq 13$	$14 \leq k \leq 15$	$k \geq 16$

k-secant via incidence variety

For d-uple Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$, we regard the incidence variety $I \subset \mathbb{P}^N \times (\mathbb{P}^n)^k$ to be the (Zariski) closure of

$$I^0 := \{(a, x_1', \ldots, x_k') \mid a \in \langle\, x_1, \ldots, x_k \,\rangle \text{ and } \dim\langle\, x_1, \ldots, x_k \,\rangle = k-1\}\,,$$

where we write $x_i = \nu_d(x_i')$ for $x_i' \in \mathbb{P}^n$.

$$\begin{tikzcd}
 & \mathbb{P}^N \times (\mathbb{P}^n)^k \supset I = I^0 \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[drr, bend left, "q_i"] & & \\
\sigma_k(\nu_d(\mathbb{P}^n)) \subset \mathbb{P}^N & & (\mathbb{P}^n)^k \arrow[r, "pr_i"'] & \mathbb{P}^n
\end{tikzcd}$$

We have $p(I) = \sigma_k(\nu_d(\mathbb{P}^n))$. and $\dim I = nk + k - 1$ by considering general fibers of $q : I \to (\mathbb{P}^n)^{(k)}$.

Main idea

Let $Z, X \subset \mathbb{P}^N$ be projective varieties of dimensions m, n. Suppose $Z \subset X$ and $Z \subset \mathbb{P}^\beta$, where $\mathbb{P}^\beta$ is a β-plane of $\mathbb{P}^N$ (i.e. Z is degenerate). Let's also consider a similar incidence variety $I \subset \mathbb{P}^\beta \times (Z)^k$ for Z.

$\Rightarrow$ in our case, $X = \nu_d(\mathbb{P}^n), Z = \nu_d(\mathbb{P}^m)$.

Main idea

$$F \cup \circledast \cdots \cup \circledast = p^{-1}(a) \subset I \subset \mathbb{P}^\beta \times (Z)^k$$

$\uparrow$ an irred. comp. of $p^{-1}(a)$

$$I \xrightarrow{p} \mathbb{P}^\beta \ni a \text{ (gen'l)}, \qquad I \xrightarrow{q} (Z)^k, \qquad I \xrightarrow{q_i} Z$$

$$(Z)^k \longrightarrow Z \xleftarrow[\simeq]{\nu_d} \mathbb{P}^m$$

Let $Z, X \subset \mathbb{P}^N$ be projective varieties of dimensions m, n. Suppose $Z \subset X$ and $Z \subset \mathbb{P}^\beta$, where $\mathbb{P}^\beta$ is a β-plane of $\mathbb{P}^N$ (i.e. Z is degenerate). Let's also consider a similar incidence variety $I \subset \mathbb{P}^\beta \times (Z)^k$ for Z.

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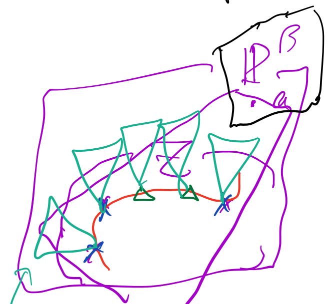
$\Pi_a \sigma_k(\nu_d \mathbb{P}^n)$



$\Pi_{x_i} \sigma_k(\nu_d \mathbb{P}^n)$

① if the secant fiber $p^{-1}(a)$ has positive dim. (e.g. $k \cdot m + k - 1 > \beta$ or $\sigma_k(Z)$ is defective) $\Longrightarrow$

or ② if a is not gen. identifiable

$\Longrightarrow$ $\exists$ also a positive dim'l k-tangential contact locus contracting $\Pi_a \sigma_k(\nu_d \mathbb{P}^n)$

Tangents on subsecant loci

Now, we estimate the dimension of span of moving tangents.

As changing homogeneous coordinates $t_0, t_1, \dots, t_m, u_1, u_2, \dots, u_{m'}$ on $\mathbb{P}^n$ with $m' := n - m$, we may assume that $\mathbb{P}^m$ is the zero set of $u_1, \dots, u_{m'}$ and that on the affine open subset $\{t_0 \neq 0\}$, the Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is parametrized by monomials of $\mathbb{C}[t_1, \dots, t_m, u_1, \dots, u_{m'}]$ of degree $\leq d$. Let $x = \nu_d(x')$ with $x' \in \{t_0 \neq 0\} \subset \mathbb{P}^n$. Then $\mathbb{T}_x \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ coincides with

$$\begin{bmatrix} \nu_d \\ \partial\nu_d/\partial t_1 \\ \vdots \\ \partial\nu_d/\partial t_m \\ \partial\nu_d/\partial u_1 \\ \vdots \\ \partial\nu_d/\partial u_{m'} \end{bmatrix}(x') = \begin{bmatrix} \mathbf{mono}[t]_{\leq d} & u_1 \cdot \mathbf{mono}[t]_{\leq d-1} & \cdots & u_{m'} \cdot \mathbf{mono}[t]_{\leq d-1} & * \\ (\mathbf{mono}[t]_{\leq d})_{t_1} & u_1 \cdot (\mathbf{mono}[t]_{\leq d-1})_{t_1} & \cdots & u_{m'} \cdot (\mathbf{mono}[t]_{\leq d-1})_{t_1} & * \\ \vdots & \vdots & & \vdots & \vdots \\ (\mathbf{mono}[t]_{\leq d})_{t_m} & u_1 \cdot (\mathbf{mono}[t]_{\leq d-1})_{t_m} & \cdots & u_{m'} \cdot (\mathbf{mono}[t]_{\leq d-1})_{t_m} & * \\ \mathbf{O} & \mathbf{mono}[t]_{\leq d-1} & \cdots & \mathbf{O} & * \\ \vdots & & \ddots & & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{mono}[t]_{\leq d-1} & * \end{bmatrix}(x')$$

, where $\text{mono}[t]_{\leq e}$ denotes the set of monomials $f \in \mathbb{C}[t_1, \dots, t_m]$ with $\deg f \leq e$ and $(\text{mono}[t]_{\leq e})_{t_i}$ $\{\partial f/\partial t_i \mid f \in \text{mono}[t]_{\leq e}\}$ and O is a zero matrix with suitable size.

Tangents on subsecant loci

In particular, in case of $x' \in \mathbb{P}^m = \{u_1 = \cdots = u_{m'} = 0\}$, we see that the matrix is of the form

$$\begin{bmatrix} \textbf{mono}[t]_{\leq d} & \mathbf{o} & \cdots & \mathbf{o} & \mathbf{o} \\ (\textbf{mono}[t]_{\leq d})t_1 & \mathbf{o} & \cdots & \mathbf{o} & \mathbf{o} \\ \vdots & \vdots & & \vdots & \vdots \\ (\textbf{mono}[t]_{\leq d})t_m & \mathbf{o} & \cdots & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \textbf{mono}[t]_{\leq d-1} & \cdots & \mathbf{o} & \mathbf{o} \\ \vdots & & \ddots & & \vdots \\ \mathbf{o} & \mathbf{o} & \cdots & \textbf{mono}[t]_{\leq d-1} & \mathbf{o} \end{bmatrix} (x') \ .$$

As a consequence, we derive a key proposition

Proposition I (estimate on span of moving embedded tangents)

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple Veronese embedding. For an m-plane $\mathbb{P}^m \subset \mathbb{P}^n$, for a (possibly reducible) subset $A \subset \mathbb{P}^m$, and for a linear variety $\Lambda \subset \langle\, \nu_d(\mathbb{P}^m)\,\rangle$,

$$\dim\langle\, \Lambda \cup \bigcup_{x \in \nu_d(A)} \mathbb{T}_x \nu_d(\mathbb{P}^n)\,\rangle \geq \dim\langle\, \Lambda \cup \nu_d(A)\,\rangle + (n-m)\big(1 + \dim\langle\, \nu_{d-1,m}(A)\,\rangle\big) \cdots (*)$$

, where $v_{e,m} : \mathbb{P}^m \to \mathbb{P}^{\binom{m+e}{e}-1}$ is the e-th Veronese embedding of $\mathbb{P}^m$.

Tangents on subsecant loci

Another key proposition is

Proposition II

Let $Z = \nu_d(\mathbb{P}^m) \subset \mathbb{P}^\beta$ *with* $d \geq 3$, $2 \leq m \leq k-2$, *and* $\beta = \binom{m+d}{m} - 1$. *Assume that* $\dim(\sigma_{k-1}(Z)) = (k-1)m + k - 2 < \beta$. *Let* $I \subset \mathbb{P}^\beta \times (Z)^k$ *be the previous incidence,* $(a, x_1, \ldots, x_k) \in I$ *be a general point, and let* $F \subset I$ *be an irreducible component of* $p^{-1}(a)$ *containing* $(a, x_1, \ldots, x_k)$. *For the preimage* $A \subset \mathbb{P}^m$ *of* $q_i(F) \cup \{x_1, \ldots, x_k\} \subset Z$ *under* $\nu_d : \mathbb{P}^m \simeq Z$, *if* $(d, m) \neq (3, 2)$, *then we have*

$$\dim\langle\, \nu_{d-1,m}(A)\,\rangle \geq k - 1 + (km + k - 1) - \dim \sigma_k(Z)$$

for each i *with* $1 \leq i \leq k$.

Tangents on subsecant loci

Another key proposition is

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Let $Z = \nu_d(\mathbb{P}^m) \subset \mathbb{P}^\beta$ *with* $d \geq 3$, $2 \leq m \leq k-2$, *and* $\beta = \binom{m+d}{m} - 1$. *Assume that* $\dim(\sigma_{k-1}(Z)) = (k-1)m + k - 2 < \beta$. *Let* $I \subset \mathbb{P}^\beta \times (Z)^k$ *be the previous incidence,* $(a, x_1, \ldots, x_k) \in I$ *be a general point, and let* $F \subset I$ *be an irreducible component of* $p^{-1}(a)$ *containing* $(a, x_1, \ldots, x_k)$. *For the preimage* $A \subset \mathbb{P}^m$ *of* $q_i(F) \cup \{x_1, \ldots, x_k\} \subset Z$ *under* $\nu_d : \mathbb{P}^m \simeq Z$, *if* $(d, m) \neq (3, 2)$, *then we have*

$$\dim\langle\, \nu_{d-1,m}(A)\,\rangle \geq k - 1 + (km + k - 1) - \dim \sigma_k(Z)$$

for each i *with* $1 \leq i \leq k$.

Note that we can improve the low bound for $\dim\langle\, \nu_{d-1,m}(A)\,\rangle$ according to the situation of (k, d, m).

Tangents on subsecant loci

Proof of Proposition II) Set $s = \dim \sigma_k(Z)$. Then, in our situation $\dim q_1(F) = (km + k - 1) - s$.

Let $q_1(F)' \subset \mathbb{P}^m$ be the preimage of $q_1(F) \subset Z$ under $\nu_d : \mathbb{P}^m \simeq Z$, and let $A = q_1(F)' \cup \{x_1', \ldots, x_k'\} \subset \mathbb{P}^m$. Let $\nu_{d-1} = \nu_{d-1,m} : \mathbb{P}^m \to \mathbb{P}^{\beta_{d-1}}$ be the $(d-1)$-uple Veronese embedding with $\beta_{d-1} = \binom{m+d-1}{m} - 1$. Then the $(k-1)$-plane $\langle\, \nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\,\rangle$ is contained in

$$\langle\, \nu_{d-1}(A)\,\rangle = \langle\, \nu_{d-1}(q_1(F)') \cup \{\nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\}\,\rangle$$

and is of codimension $e = \dim\langle\, \nu_{d-1}(A)\,\rangle - (k-1)$. We can also show $\mathrm{codim}(\nu_{d-1}(\mathbb{P}^m), \mathbb{P}^{\beta_{d-1}}) \geq k$. So, by a generalization of Trisecant lemma,

$$\nu_{d-1}(\mathbb{P}^m) \cap \langle\, \nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\,\rangle = \{\nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\}\,.$$

In particular, $\dim(\nu_{d-1}(q_1(F)') \cap \langle\, \nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\,\rangle) = 0$. On the other hands, in $\langle\, \nu_{d-1}(A)\,\rangle$, we have

$$0 = \dim(\nu_{d-1}(q_1(F)') \cap \langle\, \nu_{d-1}(x_1'), \ldots, \nu_{d-1}(x_k')\,\rangle) \geq \dim(\nu_{d-1}(q_1(F)')) - e.$$

Hence
$\dim(\langle\, \nu_{d-1}(A)\,\rangle) \geq k - 1 + \dim(\nu_{d-1}(q_1(F)')) = k - 1 + (km + k - 1) - s.$

Idea of proof for non-trivial Singularity (ii)

Basically, part (ii) corresponds to $km+k-1>\beta$, $\sigma_k(\nu_d(\mathbb{P}^m))=\mathbb{P}^\beta$, and $\dim(\sigma_{k-1}(\nu_d(\mathbb{P}^m)))=(k-1)m+k-2<\beta$.
Now, suppose that $\sigma_k(\nu_d(\mathbb{P}^m))\not\subset \mathrm{Sing}(\sigma_k(\nu_d(\mathbb{P}^n)))$. Take a general point $(a,x_1,\dots,x_k)\in I$ and an irreducible component F of $p^{-1}(a)$ containing $(a,x_1,\dots,x_k)$. Then $a\in\sigma_k(\nu_d(\mathbb{P}^m))$ be a general (so, smooth) point. By Terracini's lemma, we have

$$\langle \bigcup_{x\in q_i(F)\cup\{x_1,\dots,x_k\}} \mathbb{T}_x(\nu_d(\mathbb{P}^n))\ \rangle \subset \mathbb{T}_a(\sigma_k(\nu_d(\mathbb{P}^n))). \tag{1}$$

We take $A=\nu_d^{-1}(q_i(F)\cup\{x_1,\dots,x_k\})$. By Propositions I and II to this A and $\Lambda=\langle\,\nu_d(\mathbb{P}^m)\,\rangle=\mathbb{P}^\beta$, the number $(*)$ is less than or equal to $\beta+(n-m)(k+(km+k-1)-\beta)$. From (1), we have

$$\beta+(n-m)(k+(km+k-1)-\beta)\le kn+k-1.$$

Hence $(n-m)((km+k-1)-\beta)\le(km+k-1)-\beta$. It contradicts the assumption $km+k-1>\beta$, for example, if $n\ge m+2$.

Singularity can happen at a special point

Let $V = \mathbb{C}\langle x, y, z, w\rangle \supset W = \mathbb{C}\langle x, y, z\rangle$ and $f = x^2y^2 + z^4$ be a form of degree 4. Then, f represents a point in $\sigma_4(\nu_4(\mathbb{P}W)) \setminus \sigma_3(\nu_4(\mathbb{P}V))$. Note that $\mathrm{rank}\phi_{2,2}(f) = 4 > 3$.

Theorem B (i) shows that a *general* form in $\sigma_4(\nu_4(\mathbb{P}W)) \setminus \sigma_3(\nu_4(\mathbb{P}V))$ is smooth. But, here we show that f is a singular point of $\sigma_4(\nu_4(\mathbb{P}V))$.

We know that the form $f_D = x^2y^2$ has Waring rank 3 so that $f_D = \ell_1^4 + \ell_2^4 + \ell_3^4$ for some $\ell_i \in \mathbb{C}[x, y]_1$. By (H. 2018) f_D is also a singular point of $\sigma_3(\nu_4(\mathbb{P}V))$. Since $f \in \langle f_D, z^4\rangle$, using similar ideas before, we see that

$$\mathbb{T}_f\sigma_4(\nu_4(\mathbb{P}V)) \supseteq \langle \bigcup_{\ell_i \in \mathbb{P}^1} \mathbb{T}_{\ell_i}\nu_4(\mathbb{P}V), \mathbb{T}_{z^4}\nu_4(\mathbb{P}V)\rangle \ ,$$

which implies $\dim \mathbb{T}_f\sigma_4(\nu_4(\mathbb{P}V)) \geq 16 = 12 + 3 + 1$, greater than the expect dimension 15.

4-th secant variety

As an application of our main results, we also consider the case of singular loci of the fourth-secant variety of any Veronese embeddings.

Theorem D

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple Veronese embedding with $n \geq 3$, $d \geq 3$, and $N = \binom{n+d}{d} - 1$. Then the followings hold.

(i) $\sigma_4(\nu_d(\mathbb{P}^n))$ *is smooth at every point which does not belong to* $\sigma_3(\nu_d(\mathbb{P}^n))$ *and any 2-subsecant* $\sigma_4(\nu_d(\mathbb{P}^2))$ *of* $\sigma_4(\nu_d(\mathbb{P}^n))$.

(ii) *A general point in* $\sigma_4(\nu_d(\mathbb{P}^2)) \setminus \sigma_3(\nu_d(\mathbb{P}^n))$ *is also a smooth point of* $\sigma_4(\nu_d(\mathbb{P}^n))$ *for* $d \geq 4$. *When* $d = 3$ *and* $n = 3$, *all points in* $\sigma_4(\nu_d(\mathbb{P}^2)) \setminus \sigma_3(\nu_d(\mathbb{P}^n))$ *are smooth. If* $d = 3$, $n \geq 4$, *then* $\sigma_4(\nu_d(\mathbb{P}^2)) \subset \mathrm{Sing}(\sigma_4(\nu_d(\mathbb{P}^n)))$ *and* $\sigma_4(\nu_d(\mathbb{P}^2)) \not\subset \sigma_3(\nu_d(\mathbb{P}^n))$ *(i.e. non-trivial singularity).*

(iii) $\sigma_4(\nu_d(\mathbb{P}^1)) \setminus \sigma_3(\nu_d(\mathbb{P}^n)) \subset \mathrm{Sm}(\sigma_4(\nu_d(\mathbb{P}^n)))$ *if* $d \geq 7$. *When* $d = 6$, $\sigma_4(\nu_d(\mathbb{P}^1)) \subset \mathrm{Sing}(\sigma_4(\nu_d(\mathbb{P}^n)))$, $\sigma_4(\nu_d(\mathbb{P}^1)) \not\subset \sigma_3(\nu_d(\mathbb{P}^n))$ *(i.e. non-trivial singularity) and* $\sigma_4(\nu_d(\mathbb{P}^1)) \subset \sigma_3(\nu_d(\mathbb{P}^n))$ *when* $d \leq 5$.

Main ingridient: Normal forms due to Landsberg-Teitler (5 types), Young flattening $YF^a_{d_1,d_2,n} : S^d V \to S^{d_1} V \otimes S^{d_2} V \otimes \bigwedge^a V^* \otimes \bigwedge^{a+1} V$

Non-singularity via Conormal space

Let $X \subset \mathbb{P}W$ be any variety. For any linear embedding $W \hookrightarrow A \otimes B$ and the induced embedding $X \subset \mathbb{P}W \hookrightarrow \mathbb{P}(A \otimes B)$, it is well-known that for any $f \in \sigma_k(X) \subset \mathbb{P}(A \otimes B)$, we have

$$\hat{N}_f^* \sigma_k(X) \supseteq \ker(f) \otimes \mathrm{im}(f)^{\perp} = \hat{N}_f^* \sigma_p(Seg(\mathbb{P}A \times \mathbb{P}B)) \quad \text{in } A^* \otimes B^*$$

provided that $X \subseteq \sigma_p(Seg(\mathbb{P}A \times \mathbb{P}B))$, $X \not\subseteq \sigma_{p-1}(Seg(\mathbb{P}A \times \mathbb{P}B))$ and f has rank $k \cdot p$ as a linear map in $\mathrm{Hom}(A^*, B)$. Further, since $X \subset \mathbb{P}W \subset \mathbb{P}(A \otimes B)$, then as a subvariety of $\mathbb{P}W$ it holds that

$$\hat{N}_f^* \sigma_k(X) \supseteq \pi(\ker(f) \otimes \mathrm{im}(f)^{\perp}) = \hat{N}_f^* (\sigma_p(Seg(\mathbb{P}A \times \mathbb{P}B)) \cap \mathbb{P}W)\ ,$$

where $\pi : A^* \otimes B^* \to W^*$ is the dual map of $W \hookrightarrow A \otimes B$.

If we apply this fact to a partial polarization $S^d V \hookrightarrow S^a V \otimes S^{d-a} V$ (see $X = \nu_d(\mathbb{P}V)$ is contained in $Seg(\mathbb{P}S^a V \times \mathbb{P}S^{d-a} V)$ in $\mathbb{P}(S^a V \otimes S^{d-a} V)$, i.e. $p = 1$ case), then

Non-singularity via Conormal space

Proposition

Suppose that $f \in S^d V$ corresponds to a (closed) point of $\sigma_k(\nu_d(\mathbb{P}V)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}V))$. Then, for any a with $1 \leq a \leq \lfloor \frac{d+1}{2} \rfloor$ with $\mathrm{rank}\phi_{d-a,a}(f) = k$ we have

$$\hat{N}^*_{[f]}\sigma_k(\nu_d(\mathbb{P}V)) \supseteq (f^\perp)_a \cdot (f^\perp)_{d-a} \tag{2}$$

as a subspace of $T_d = S^d V^*$.

Non-singularity via Conormal space

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Suppose that $f \in S^d V$ corresponds to a (closed) point of $\sigma_k(\nu_d(\mathbb{P}V)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}V))$. Then, for any a with $1 \leq a \leq \lfloor \frac{d+1}{2} \rfloor$ with $\mathrm{rank}\phi_{d-a,a}(f) = k$ we have

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as a subspace of $T_d = S^d V^*$.

If applying it to $S^d V \hookrightarrow S^{d_1} V \otimes \bigwedge^a V^* \otimes S^{d_2} V \otimes \bigwedge^{a+1} V$

Proposition

Let $V = \mathbb{C}^{n+1}$ and f be any (closed) point of $\sigma_k(\nu_d(\mathbb{P}V)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}V))$ in $\mathbb{P}S^d V$. Suppose $\mathrm{YF}^a_{d_1,d_2,n}(f)$ has rank $k\binom{n}{a}$ as a linear map in $\mathrm{Hom}(S^{d_1} V^* \otimes \bigwedge^a V, S^{d_2} V \otimes \bigwedge^{a+1} V)$. Then,

$$\hat{N}^*_f\sigma_k(\nu_d(\mathbb{P}V)) \supseteq (\ker \mathrm{YF}^a_{d_1,d_2,n}(f)) \cdot (\mathrm{im}\mathrm{YF}^a_{d_1,d_2,n}(f))^\perp \ , \tag{3}$$

where the right hand side is to be understood as the image of the multiplication $S^{d_1} V^* \otimes \bigwedge^a V \otimes S^{d_2} V^* \otimes \bigwedge^{a+1} V^* \to S^d V^*$.

Concluding Remark

As we mentioned before, each point $p \in \sigma_k(\nu_d(\mathbb{P}^n)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ is located in $\sigma_k(\nu_d(\mathbb{P}^m)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ for some $1 \leq m \leq \min\{k-1, n\}$. To make the picture more complete, we have two future issues:

(i) For the subsecant loci (i.e. $m < \min\{k-1, n\}$), one needs to check (non-)singularity not only at a general point but also at every point.

(ii) Smoothness of points in full-secant loci (i.e. $m = \min\{k-1, n\}$) should be answered.

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As we mentioned before, each point $p \in \sigma_k(\nu_d(\mathbb{P}^n)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ is located in $\sigma_k(\nu_d(\mathbb{P}^m)) \setminus \sigma_{k-1}(\nu_d(\mathbb{P}^n))$ for some $1 \leq m \leq \min\{k-1, n\}$. To make the picture more complete, we have two future issues:

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(ii) Smoothness of points in full-secant loci (i.e. $m = \min\{k-1, n\}$) should be answered.

Issue (i) is expected to be very complicated, because at a *special* point (as shown in Example) singularity can also happen (in fact, we can generate more singular examples using similar idea). For this case, in general, one could not hope to find normal forms and the situation is expected to be wild (in other words, the subsecant loci can't be covered with finitely many nice families of SL-orbits).

Concluding Remark

Issue (ii) deals with case $m = \min\{k-1, n\}$. First, suppose $n \leq k-1$. Then $m = n$.

Example Let us consider $\sigma_{14}(\nu_8(\mathbb{P}^2))$. Take general 14 points on $\nu_8(\mathbb{P}^2)$. Angelini-Chiantini (2019) showed an existence of a *non-normal* point to $\sigma_{14}(\nu_8(\mathbb{P}^2))$ in the linear span of these points (note that this may correspond to Part (i) of Theorem B if we take $n = m = 2$). Further, our previous example also fits into this case even in a small k.

Concluding Remark

Issue (ii) deals with case $m = \min\{k-1, n\}$. First, suppose $n \leq k-1$. Then $m = n$.

Example Let us consider $\sigma_{14}(\nu_8(\mathbb{P}^2))$. Take general 14 points on $\nu_8(\mathbb{P}^2)$. Angelini-Chiantini (2019) showed an existence of a *non-normal* point to $\sigma_{14}(\nu_8(\mathbb{P}^2))$ in the linear span of these points (note that this may correspond to Part (i) of Theorem B if we take $n = m = 2$). Further, our previous example also fits into this case even in a small k.

This might be happened due to a relatively low value n compared to k (i.e. $n \leq k-1$) Thus, a proper question for the full-secant loci is probably such as

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Question 1(full secant loci) Suppose $k-1 \leq n$ and let $\mathcal{D}$ be the subsecant loci of $\sigma_k(\nu_d(\mathbb{P}^n))$. Are the points of $\sigma_k(\nu_d(\mathbb{P}^{k-1})) \setminus (\mathcal{D} \cup \sigma_{k-1}(\nu_d(\mathbb{P}^n)))$ all smooth in $\sigma_k(\nu_d(\mathbb{P}^n))$?

Note that the answer to this question is affirmative in cases of $k = 2$ classically and $k = 3$ by (H. 2018) and $k = 4$ by Theorem D.

Concluding Remark

Question 2(for other nice varieties) What about higher secants of other varities (e.g. Segre, Segre-Veronese, Grassmannians, etc)? Can we do similar things using the concept of 'subsecant loci'?

For instance, the singular locus of 2-secant of Segre embedding (done by Michalek-Oeding-Zwiernik('15)) can be recovered by the same perspective.

Thank you!