On secant defectiveness and identifiability of Segre-Veronese varieties

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Geometry of secants
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Notation 1.1. Let $SV^{n_1,\ldots,n_r}_{d_1,\ldots,d_r}$ be the Segre-Veronese variety given as the image of the embedding

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^N$$

via the linear system $\left|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}}(d_1,\ldots,d_r)\right|$.

Theorem 1.2. The Segre-Veronese variety $SV_{d_1,\dots,d_r}^{n_1,\dots,n_r} \subseteq \mathbb{P}^N$ is not h-defective for

$$h \leq \frac{d_j}{n_j + d_j} \cdot \frac{1}{1 + \sum_{i=1}^r n_i} \cdot \prod_{i=1}^r \binom{n_i + d_i}{d_i},$$

where $n_j/d_j := \max\{n_i/d_i \ : \ 1 \leq i \leq r\}$. Furthermore, if in addition $2\sum_{i=1}^r n_i$ is smaller than the right hand side of the above inequality, then the Segre-Veronese variety is $(h-1)$-identifiable.

Remark 1.3. The theorem gives a polynomial bound of degree n_i in d_i and of degree $d_i - 2$ in n_i. The expected generic rank is given by a polynomial of degree n_i in d_i and of degree $d_i - 1$ in n_i. Here are some cases with three factors and $n_1 = n_2 = n_3$:

n_1	d_1	d_2	d_3	Theorem 1.2	[AMR19, Thm. 4.8]	[CGG05, Prop. 3.2]
2	3	3	3	$h \leq 85$	$h \leq 19$	$h \leq 3$
2	3	4	4	$h \leq 193$	$h \leq 21$	$h \leq 3$
2	3	5	5	$h \leq 378$	$h \leq 25$	$h \leq 3$
3	5	5	5	$h \leq 10976$	$h \leq 64$	$h \leq 4$
10	5	6	6	$h \leq 2070715873$	$h \leq 13311$	$h \leq 11$
30	5	5	7	$h \leq 1703293480928730$	$h \leq 893731$	$h \leq 31$

Remark 1.4.

- Secant defectiveness for Segre-Veronese products has been solved in some very special cases [CGG05], [AB09], [Abo10], [BCC11], [AB12], [BBC12], [AB13]. It has been completely solved for products $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, in [LP13].
- Identifiability of Segre-Veronese varieties has been recently given in [FCM20], and in [BBC18] under hypotheses of non secant defectiveness. In general, h-defectiveness of Segre products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \subseteq \mathbb{P}^N$ is classified only for $h \leq 6$ [AOP09].

Let $N \simeq \mathbb{Z}^n$, let $M := \mathrm{Hom}(N, \mathbb{Z})$ be its dual and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. A full-dimensional lattice polytope $P \subseteq M_{\mathbb{Q}}$ defines a (not necessarily normal) toric variety X_P which is the Zariski closure of the image of the monomial map

$$\begin{array}{rccc} \phi_P : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{P}^{|P\cap M|-1} \\ & u & \mapsto & [u^m : m \in P \cap M]. \end{array}$$

Each element $v \in N$ defines a 1-parameter subgroup $t \mapsto t^v$ of the torus whose Zariski closure in X_P is the curve Γ_v.

Observation 1.5. Given a point $a \in \mathbb{C}^*$ the tangent space of X_P at $\Gamma_v(a)$ is the projectivization of the vector subspace generated by the rows of the following matrix

$$M_v(a) := \begin{pmatrix} a^{\langle m_1, v\rangle} & \dots & a^{\langle m_r, v\rangle} \\ \langle m_1, e_1\rangle a^{\langle m_1 - e_1^*, v\rangle} & \dots & \langle m_r, e_1\rangle a^{\langle m_r - e_1^*, v\rangle} \\ \vdots & & \vdots \\ \langle m_1, e_n\rangle a^{\langle m_1 - e_n^*, v\rangle} & \dots & \langle m_r, e_n\rangle a^{\langle m_r - e_n^*, v\rangle} \end{pmatrix}$$

where $P \cap M = \{m_1, \dots, m_r\}$.

Notation 1.6. The matrix $M_{v_1,\dots,v_h}(a)$ is the vertical join of $M_{v_1}(a), \dots, M_{v_h}(a)$.

Observation 1.7. By Terracini's lemma the linear span

$$\Lambda_{v_1,\dots,v_h}(a) := T_{\Gamma_{v_1}(a)}X_P + \cdots + T_{\Gamma_{v_h}(a)}X_P$$

of the tangent spaces of X_P at the (general) points $\Gamma_{v_1}(a), \dots, \Gamma_{v_h}(a)$ is the tangent space of $\sigma_h(X_P)$ at a general point of the $(h-1)$-secant plane generated by these points. Then

$$\operatorname{rk} M_{v_1,\dots,v_h}(a) \text{ is maximal} \Rightarrow \dim \sigma_h(X_P) = \operatorname{edim} \sigma_h(X_P).$$

Definition 1.8. A *simplex* is a subset $\Delta \subseteq M \simeq \mathbb{Z}^n$ of cardinality $n+1$ not contained in an affine hyperplane of $M_{\mathbb{Q}}$. Given a subset $S \subseteq M$ denote by

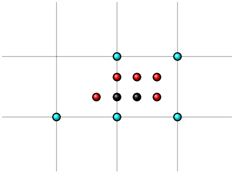
$$\mathscr{S}(S) := \{\Delta : \Delta \subseteq S \text{ and } \Delta \text{ is a simplex}\}.$$

Given a simplex Δ, denote by $b(\Delta)$ its barycenter. We say that $v \in N$ *separates the simplex* Δ in a subset $S \subseteq M$ if the maximum of the function

$$\mathscr{S}(S) \to \mathbb{Z} \qquad \Delta \mapsto \langle b(\Delta), v \rangle$$

is attained exactly at Δ.

Example 1.9. Let S be the subset of the plane given by the blue points. The red points are unique barycenters of simplexes contained in S, while black dots are repeated barycenters.



It is possible to prove that each vertex of the *barycentric polytope*, the convex hull of $\{b(\Delta) : \Delta \in \mathscr{S}(S)\}$, comes from exactly one simplex. The simplex corresponding to the top-right vertex of the barycentric polytope is separated by $v = (1, 1)$.

Proposition 1.10. Let S be a subset of $P \cap M$, with $M \simeq \mathbb{Z}^n$, and assume the following.

1. There are disjoint simplices $\Delta_1, \ldots, \Delta_h \subseteq S$.
2. There are $v_1, \ldots, v_h \in N$ such that v_i separates the simplex Δ_i in $S \setminus \Delta_1 \cup \cdots \cup \Delta_{i-1}$, for each $1 \leq i \leq h$.

Then, up to a rescaling of the v_i and of a, the matrix $M_{v_1,\ldots,v_h}(a)$ has maximal rank $(n+1)h$.

Idea of proof.

- Given a simplex $\Delta := \{m_{i_0}, \ldots, m_{i_n}\}$ the corresponding $(n+1) \times (n+1)$ minor of the matrix $M_v(a)$ (multiplying its rows by non-zero constants) is

$$\delta_{v,\Delta}(a) := \frac{a^{\langle m_{i_0}+\cdots+m_{i_n}, v\rangle}}{a^{\langle e_1^*+\cdots+e_n^*, v\rangle}} \begin{vmatrix} 1 & \ldots & 1 \\ \langle m_{i_0}, e_1\rangle & \ldots & \langle m_{i_n}, e_1\rangle \\ \vdots & & \vdots \\ \langle m_{i_0}, e_n\rangle & \ldots & \langle m_{i_n}, e_n\rangle \end{vmatrix}$$

  Observe that $\delta_{v,\Delta}(a)$ is non-zero exactly when Δ is a simplex. Up to rescaling $M_v(a)$ the exponent of a is $\langle b(\Delta), v\rangle$.

- Given $K, L \subseteq [s] := \{1, 2, \ldots, s\}$ and an $s \times s$ matrix A we denote by $A_{K,L}$ the determinant of the submatrix obtained from A whose rows and columns are indexed by the set K and L respectively. Given a set of rows J, with $|J| = n+1$, then [Jan08]

$$\det(A) = \sum_{I \subseteq [s]\,, |I| = n+1} (-1)^{\varepsilon(J,I)} A_{J,I} A_{J^c,I^c},$$

where $\varepsilon(J, I) \in \mathbb{Z}$ depends only on I and J.

- Let M be the $(n+1)h \times (n+1)h$ square submatrix of the rescaled $M_{v_1,\dots,v_h}(a)$ whose columns correspond to the points of $\Delta_1 \cup \cdots \cup \Delta_h$.
- Applying the above Laplace's expansion several times we can write this determinant as follows:

$$\det(M) = \sum_{(I_1,\dots,I_h)\in\mathcal{P}(n+1,h)} (-1)^{\varepsilon(I_1,\dots,I_h)} M_{I_1} M_{I_2} \cdots M_{I_h}$$

  where M_{I_j} is the determinant of the $(n+1)\times(n+1)$ submatrix of M whose columns and rows are labeled respectively by I_j and $\{(j-1)(n+1)+1,\dots,j(n+1)\}$.

- One of its terms is the non-zero product

$$\delta_{v_1,\Delta_1}(a)\cdots\delta_{v_h,\Delta_h}(a).$$

  Moreover, observe that each term of the determinant has the above form for some partition of S into k disjoint subsets of cardinality $n+1$.
- We prove that, up to rescaling the vectors $v_1,\ldots,v_h$, the above product is the leading term of the determinant and thus the matrix has maximal rank. □

Instead of computing the barycentric polytope (too slow) we apply the following algorithm.

Input : a finite, full-dimensional subset $S \subseteq M$
while S *is full-dimensional* **do**
  choose $v \in N_{\mathbb{Q}}$ such that $x \mapsto \langle v, x \rangle$ is injective on S;
  reorder S increasingly according to these values;
  define $\Delta := \{\max(S)\}$;
  repeat
    $x := \max\{u \in S \setminus \Delta \,:\, \Delta \cup \{u\} \text{ is independent}\}$;
    $\Delta := \Delta \cup \{x\}$;
  until Δ *is full-dimensional*;
  $S := S \setminus \Delta$;
end
if S *is independent* **then** **return** false ;
else **return** true ;

Observation 1.11. It is possible to show that the above algorithm produces a vertex of the barycentric polytope where the linear function $\langle v, \_\rangle$ attains its maximum. A magma implementation of the algorithm is given here:

https://github.com/alaface/secant-algorithm

Theorem 1.12. Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope and $X_P \subseteq \mathbb{P}^{|P \cap M|-1}$ the corresponding n-dimensional toric variety. If

$$h \leq \frac{|P \cap M| - \max\{|H \cap P \cap M| \,:\, H \text{ is a hyperplane}\}}{n+1}$$

then X_P is not h-defective. If moreover X_P is smooth and $2\dim(X_P)$ is smaller than the right hand side of the above inequality then X_P is $(h-1)$-identifiable.

Observation 1.13.

- The first part of the theorem is an immediate consequence of our algorithm, since if a set of points is not contained in a hyperplane, it is always possible to find a direction which separates a symplex.
- This part of the theorem can also be proved by using tropical approach to secant defectivity [Dra08].
- The second part of the theorem is consequence of [CM22, Thm. 3] which states that if $X \subseteq \mathbb{P}^N$ is smooth, non h-defective and $2\dim(X) < h$, then X is $(h-1)$-identifiable.

Proposition 1.14. Let $P \subseteq M_{\mathbb{Q}} \simeq \mathbb{Q}^n$ be a full-dimensional lattice polytope such that there exist linearly independent $v_1, \ldots, v_n \in N$ and facets $F_1, \ldots, F_n$ such that for any i we have

$$v_j(F_i) = v_j(P) \quad \text{for any } j \neq i.$$

Then, given a hyperplane $H \subseteq M_{\mathbb{Q}}$, there exists a facet F_i, with $1 \leq i \leq n$, such that $|H \cap P \cap M| \leq |F_i \cap M|$.

Idea of the proof. The projection $\pi_i \colon \mathbb{Q}^n \to \mathbb{Q}^{n-1}$ defined by $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$, satisfies $\pi_i(F_i) = \pi_i(P)$. Moreover there is one such π_i which is injective on H. □

Proof of Theorem 1.2. The Segre-Veronese variety is toric, defined by the polytope

$$P = \Delta_{d_1}^{n_1} \times \cdots \times \Delta_{d_r}^{n_r},$$

where each $\Delta_{d_i}^{n_i}$ is a standard simplex. Observe that P satisfies the conditions in Proposition 1.14.

The polytope P has

$$\prod_{i=1}^{r}\binom{d_i+n_i}{d_i}$$

integer points, and each facet is given by the Cartesian product of a facet of one of the $\Delta^{n_j}_{d_j}$ and the remaining $\Delta^{n_i}_{d_i}$ for $i \neq j$. Therefore, each facet contains

$$f_j = \binom{d_j+n_j-1}{d_j}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i}$$

points for some j.

Now, we compare the number of integer points on each facet:

$$
\begin{gathered}
f_j \leq f_k \\
\binom{d_j+n_j-1}{d_j}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i} \leq \binom{d_k+n_k-1}{d_k}\prod_{i\neq k}^{r}\binom{d_i+n_i}{d_i} \\
\binom{d_j+n_j-1}{d_j}\binom{d_k+n_k}{d_k} \leq \binom{d_k+n_k-1}{d_k}\binom{d_j+n_j}{d_j} \\
\frac{d_k+n_k}{n_k} \leq \frac{d_j+n_j}{n_j} \\
\frac{d_k}{n_k} \leq \frac{d_j}{n_j}
\end{gathered}
$$

Therefore the facet with maximum number of integer points is the one which maximizes n_i/d_i. Assume n_j/d_j is this maximum.

Then, by Theorem 1.12, we have that $\sigma_h(X_P)$ is not h-defective for any h at most the following

$$\frac{1}{1+\sum_i n_i} \cdot \left(\prod_{i=1}^{r} \binom{d_i+n_i}{d_i} - \binom{d_j+n_j-1}{d_j} \prod_{i\neq j}^{r} \binom{d_i+n_i}{d_i} \right)$$

$$= \frac{1}{1+\sum_i n_i} \cdot \binom{d_j+n_j-1}{d_j-1} \cdot \prod_{i\neq j}^{r} \binom{d_i+n_i}{d_i}$$

$$= \frac{1}{1+\sum_i n_i} \cdot \frac{d_j}{d_j+n_j} \cdot \prod_{i=1}^{r} \binom{d_i+n_i}{d_i}$$

$$= \frac{1}{1+\frac{n_j}{d_j}} \cdot \frac{1}{1+\sum_i n_i} \cdot \prod_{i=1}^{r} \binom{d_i+n_i}{d_i}.$$

□

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