Secants, Compositions, Identifiability and Apolarity in Geometry and Algebra (SCIAGA)

Jarosław Buczyński   Luca Chiantini

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addresses:
J. Buczyński, `jabu@mimuw.edu.pl`, Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland
L. Chiantini, `luca.chiantini@unisi.it`, University of Siena, Italy

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Abstract

These notes contain some preliminary material for workshop "Geometry of Secants" taking place at IMPAN in Warsaw (24–28 October 2022). It is a "cheat sheet" for participants, containing several concepts that are going to be freely used by speakers during the workshop. We explain the definitions of secant varieties, expected dimensions, apolarity action, Hilbert function, and various notions of rank. We discuss the concept of identifiability in the context of secant varieties, and phrase some elementary properties of these notions, including Terracini's Lemma on tangent space to secant variety at a general point.

1 Base field, projective space, and embedded projective variety

In these notes we focus our attention on the base field of complex numbers $\mathbb{C}$. Other base fields are sometimes considered in the topic of secant varieties, and several technical issues may arise. If any of those will appear on the workshop, they will be duly explained during the presentation.

For a finite dimensional vector space V, the dual space is denoted by $V^* = \mathrm{Hom}(V, \mathbb{C})$. Some authors use also $V^\vee$ to denote the dual space. $\mathbb{P}(V)$ will denote the naive projectivisation of a vector space V. That is, $\mathbb{C}$-points of $\mathbb{P}(V)$ are in 1 to 1 correspondence with lines in V through 0. Some authors use dual notation, called Grothendieck notation, where by $\mathbb{P}(V)$ the authors mean $\mathrm{Proj} \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d V$, which is the same as $\mathbb{P}(V^*)$ in the naive notation.

If $\dim V = n+1$, then $\mathbb{P}(V)$ is n-dimensional and whenever the possible additional structure of V is not relevant to the considered problem, we often write $\mathbb{P}(V)$ simply as $\mathbb{P}^n$.

A large part of workshop will concern various embedded projective varieties $X \subset \mathbb{P}(V)$. In particular, construction of the secant variety of X described in Section 2 heavily depends on both the variety X and its embedding into $\mathbb{P}(V)$, while the latter is often omitted in the notation. The **affine cone over** X (strictly speaking, of $X \subset \mathbb{P}(V)$) is the subvariety $\hat{X} \subset V$ obtained as the union of lines x for $x \in X$.

A **linear subspace** of $\mathbb{P}(V)$ is a projectivisation of a linear subspace $W \subset V$, or equivalently, a subvariety $\mathbb{P}(W)$ of $\mathbb{P}(V)$ such that its affine cone is a linear subspace of V. For a subset, subvariety or subscheme X of $\mathbb{P}(V)$, the **linear span** of X is denoted $\langle X \rangle$ and it is the smallest linear subspace of $\mathbb{P}(V)$ containing X (as subset, subvariety or subscheme, respectively). Note that of $X \subset \mathbb{P}(V)$ is a subvariety (or reduced scheme), then

$$\langle X(\mathbb{C}) \rangle = \langle X \rangle = \mathbb{P}\left(\operatorname{span}(\hat{X})\right),$$

where $X(\mathbb{C})$ is the set of complex points of X, and $\operatorname{span}(\hat{X})$ is the linear span (in the vector space V) of the affine cone over X. On the other hand, for a nonreduced subscheme $X \subset \mathbb{P}(V)$ we have $\langle X(\mathbb{C}) \rangle \subset \langle X \rangle$, but the equality does not necessarily hold.

Some classical (sub)varieties that might be considered during the workshop include the following examples.

Example 1.1 (Veronese variety). Fix a vector space V and an integer d. Consider the map of vector spaces:

$$\begin{aligned} V &\to \operatorname{Sym}^d V \\ v &\mapsto v^d. \end{aligned}$$

That is, v is mapped to the symmetric tensor $v \otimes v \otimes \cdots \otimes v$ of order d, viewed as a symmetric tensor in $\operatorname{Sym}^d V$. This map descends to a regular map of projective spaces $\nu_d \colon \mathbb{P}(V) \to \mathbb{P}(\operatorname{Sym}^d V)$, which is called the d-th **Veronese embedding**. The image of ν_d is called a **Veronese variety**. Again, abstractly, $\nu_d(\mathbb{P}(V))$ is isomorphic to $\mathbb{P}(V)$, but we consider $\nu_d(\mathbb{P}(V))$ as a subvariety in $\mathbb{P}(\operatorname{Sym}^d V)$, and the embedding is the source of intriguing properties of this variety.

Other (equivalent) descriptions of the d-th Veronese embedding are:

- in coordinates, after chosing the correct basis of $\operatorname{Sym}^d V$, the map is given by:

  $$[\alpha_0, \ldots, \alpha_n] \mapsto [\alpha_0^d, \alpha_0^{d-1}\alpha_1, \ldots, \alpha_n^d],$$

  where the resulting coordinates run through all possible monomials of degree d in the α_i's,

- in terms of linear systems, the d-th Veronese embedding is given by the complete linear system of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(d)$.

- in terms of homogeneous polynomials $\nu_d \colon \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_1) \to \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)$ is given by $\ell \mapsto \ell^d$.

Example 1.2 (Segre variety). Suppose $V_1, \dots, V_d$ are finite dimensional vector spaces. The cartesian product $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_d)$ admits a natural embedding into $\mathbb{P}(V_1 \otimes \cdots \otimes V_d)$ via:

$$([v_1], \dots, [v_d]) \mapsto [v_1 \otimes \cdots \otimes v_d].$$

This embedding is called the **Segre embedding** and the image is called **Segre variety**.

Other (equivalent) descriptions of the Segre embedding are:

- in coordinates,the map is given by:

$$[\alpha_{1,0}, \dots, \alpha_{1,n_1}], \dots, [\alpha_{d,0}, \dots, \alpha_{d,n_d}] \mapsto \\ [\alpha_{1,0} \cdots \alpha_{d,0},\ \alpha_{1,0} \cdots \alpha_{d-1,0}\alpha_{d,1}, \dots,\ \alpha_{1,n_1} \cdots \alpha_{d,n_d}],$$

  where the resulting coordinates run through all possible monomials of the form $\alpha_{1,i_1}\alpha_{2,i_2} \cdots \alpha_{d,i_d}$,

- in terms of linear systems, the Segre embedding is given by the complete linear system of the line bundle $\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}(V_d)}(1)$.

Example 1.3 (Grassmannian variety). Fo a vector space V of dimension $n+1$, by $Gr(k, V)$ or $Gr(k, n+1)$ we denote the **Grassmannian** parametrising k dimensional linear subspaces of V. Other possible notations include $G(k, V)$, $G(k, n+1)$, $Gr_k(n+1)$, $Gr_{\mathbb{P}^{k-1}}, \mathbb{P}(V))$, or using dimensions of projective spaces (instead of dimensions of linear spaces) like $Gr(k-1, n)$, and similar. Grassmannian admits a natural embedding, called **Plücker embedding** into $\mathbb{P}(\bigwedge^k V)$, given by:

$$Gr(k, V) \ni \langle x_1, \dots, x_k \rangle \mapsto [x_1 \wedge \cdots \wedge x_k] \in \mathbb{P}(\bigwedge^k V).$$

For a projective subscheme $X \subset \mathbb{P}(V)$, its **homogeneous ideal** in $\mathbb{C}[V]$ is the ideal of all forms vanishing on $\hat{X}$. The notation for this ideal varies greatly, depending on the context, including $I(X)$, I_X, $\mathcal{I}_X$, sometimes J_X etc.

The **Hilbert function** of a homogeneous ideal $I \subset \mathbb{C}[V]$ or of $\mathbb{C}[V]/I$ is $H_I \colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined as $H_I(d) = \dim(\mathbb{C}[V]/I)_d$, where $(\mathbb{C}[V]/I)_d$ is the d-th homogeneous part of the algebra $\mathbb{C}[V]/I$. Again, the notation for the Hilbert function may vary, and it is often used in the context of a subscheme $X \subset \mathbb{P}(V)$, where $H_X := H_{I(X)}$. For all $d < 0$ we have $H_I(d) = 0$, if $d = 0$ either $I = (1)$ and $H_I(0) = 0$ or $I \subsetneqq \mathbb{C}[V]$ and $H_I(0) = 1$. For $d \geqslant 1$, we always have:

$$H_X(d) = \dim \langle \nu_d(X) \rangle + 1.$$

2 Secant variety

Fix an embedded projective variety $X \subset \mathbb{P}(V)$.

The secant variety of X is the Zariski closure of the union of linear spans of tuples of points on X. In this note we will use the following notation:

$$\sigma_r(X) := \overline{\bigcup \{\langle x_1, \dots, x_r \rangle \mid x_i \in X\}},$$

and call the resulting variety the **r-th secant variety of X**. Many notations for secant varieties are used in the literature, and it has been decided many years ago that a uniformisation in this matter is impossible to achieve. Instead, there is a mutual tolerance for the diversity of the notation. Some other of the main notations include $S_r(X)$, and $\mathrm{Sec}_r(X)$, $\mathbb{S}\mathrm{ec}_r(X)$. To explain another option, note that for reasonable values of r and generic choices of points x_i, we have $\langle x_1, \ldots, x_r\rangle \simeq \mathbb{P}^{r-1}$, and therefore some authors prefer to call the same variety the $(r-1)$-st secant variety, denoted $S_{r-1}(X)$ or $\mathrm{Sec}_{r-1}(X)$. If you are using this other notation, then consequently, the notation for abstract secant variety (below) will be different.

If you are not interested in "higher secant varieties", then another possible notation is to call $\sigma_2(X)$ simply the secant variety of X (without reference to the number 2), and denote it $\sigma(X)$ or $S(X)$ or $\mathrm{Sec}(X)$.

If the underlying variety X is obvious from the context of the notation, then X might be omitted from the notation, for example σ_r or S_r etc.

The **abstract secant variety** is:

$$\mathbb{S}_r(X) := \overline{\{(x_1, \ldots, x_r, p) \in X^{\times r} \times \mathbb{P}(V) \mid p \in \langle x_1, \ldots, x_r\rangle\}}.$$

There is no really standard notation for the abstract secant variety, and all authors use a notation as they find convenient, impossible to list all of them. Perhaps one way, would be to use the notation of the secant variety and add a letter A in front, for instance $A\sigma_r(X)$ or $\mathrm{ASec}_{r-1}(X)$, and so on.

We always have:

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_r(X) \subset \cdots \subset \langle X\rangle .$$

For some finite integer g we will have $\sigma_g(X) = \sigma_{g+1}(X)$ and at this point we also have

$$\langle X\rangle = \sigma_g(X) = \sigma_{g+1}(X) = \cdots$$

The minimal such g is called the **generic X-rank**, see Section 3. It is clear, that it suffices to take g at most $\dim \langle X\rangle + 1$, but in fact the minimal g is much smaller than that. Some claims below and above rely on "reasonable range of r", by which we mean that $r \leqslant \dim \langle X\rangle + 1$ (otherwise the problems of secant varieties are not at all interesting, as a single secant span fills in the whole ambient space).

So in the reasonable range of r we always have $\dim \mathbb{S}_r(X) = r \dim X + (r-1)$. Consider the natural map $\mathbb{S}_r(X) \to \mathbb{P}(V)$ which is the projection on the last coordinate. By definition, its image is $\sigma_r(X)$. Therefore,

$$\dim \sigma_r(X) \leqslant \min(r \dim X + (r-1), \dim \mathbb{P}(V)).$$

The integer $\min(r \dim X + (r-1), \dim \mathbb{P}(V))$ is called the **expected dimension** of $\sigma_r(X)$, denoted $\operatorname{expdim} \sigma_r(X)$, or similar. If $\dim \sigma_r(X) < \operatorname{expdim} \sigma_r(X)$, then we say that the secant variety is **defective** and we call the difference $\delta = r \dim X + (r-1) - \dim \sigma_r(X)$ the **defect** of $\sigma_r(X)$. The defect (when defined) is also the dimension of a general fibre of the map $\mathbb{S}_r(X) \to \sigma_r(X)$.

Proposition 2.1 (Terracini Lemma)**.** *Suppose $(x_1, \ldots, x_r, p) \in \mathbb{S}_r(X)$ is a general tuple (so that $x_1, \ldots, x_r$ are general points of X and $p \in \langle x_1, \ldots, x_r\rangle$ is also general). Consider the projective tangent spaces $\mathbb{P}\hat{T}_{x_i}X$ and $\mathbb{P}\hat{T}_p\sigma_r(X)$. Then*

$$\mathbb{P}\hat{T}_p\sigma_r(X) = \left\langle \mathbb{P}\hat{T}_{x_1}X, \ldots, \mathbb{P}\hat{T}_{x_r}X \right\rangle \qquad \text{and} \qquad \dim \sigma_r(X) = \dim \left\langle \mathbb{P}\hat{T}_{x_1}X, \ldots, \mathbb{P}\hat{T}_{x_r}X \right\rangle .$$

3 X-rank and apolarity

A subvariety or subscheme $X \subset \mathbb{P}(V)$ is called **nondegenerate**, if $\langle X \rangle = \mathbb{P}(V)$.

Let $X \subset \mathbb{P}(V)$ be a nondegenerate subvariety, and $p \in \mathbb{P}(V)$ and $\hat{p} \in V$ be a $\mathbb{C}$-points such that the projective class of $\hat{p}$ is p. Then we define:

- The X-**rank** of p or $\hat{p}$ is the least integer r such that $p \in \langle x_1, \dots, x_r \rangle$ for some $x_i \in X$. The X-rank of p or $\hat{p}$ is usually denoted $R_X(p)$, $R_X(\hat{p})$, $r_X(p)$, $r_X(\hat{p})$, $\mathrm{rk}_X(p)$, $\mathrm{rk}_X(\hat{p})$, or without indicating X if it is clear from the context.

- The X-**border rank** of p or $\hat{p}$ is the least integer r such that $p \in \sigma_r(X)$. The X-border rank of p or $\hat{p}$ is usually denoted $\underline{R}_X(p)$, $\underline{r}_X(p)$, $br_X(p)$, or without indicating X if it is clear from the context (and similar notation with p replaced by $\hat{p}$).

The generic X-rank g defined in Section 2 is the same as the X-rank of a general $p \in \mathbb{P}(V)$. For any $p \in \mathbb{P}(V)$ we have

$$\underline{R}_X(p) \leqslant R_X(p), \quad \underline{R}_X(p) \leqslant g.$$

Important cases of (border) rank include Waring rank (if X is the Veronese variety), tensor rank (if X is the Segre variety), and partially symmetric rank (if X is the Segre-Veronese variety).

The **apolarity** is a basic technique to study the X-rank and related concepts in the case of $X = \nu_d(\mathbb{P}(V)) \subset \mathbb{P}(\mathrm{Sym}^d V)$. It can be generalised to a multigraded version, which is valid for many other base varieties, but we do not discuss it here.

In order to define the apolarity (in a simplest form), we consider two graded dual polynomial rings:

$$S := \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d V = \mathbb{C}[x_0, \dots, x_n] \quad \text{and} \quad T := \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d V^* = \mathbb{C}[\alpha_0, \dots, \alpha_n].$$

For a fixed d, the homogeneous elements of degree d of the first ring correspong to points in the span of $X = \nu_d(\mathbb{P}(V))$. The second ring T is seen as the homogeneous coordinate ring of $\mathbb{P}(V)$. The apolarity action of T on S is defined by:

$$\alpha_i \lrcorner F := \frac{\partial}{\partial x_i} F,$$

and further extending the action linearly and by composition. For instance, if $\Theta = 3 + \alpha_0 - 2\alpha_1^2 + \alpha_2\alpha_3$, then

$$\Theta \lrcorner F = 3F + \frac{\partial}{\partial x_0} F - 2\frac{\partial^2}{(\partial x_1)^2} F + \frac{\partial^2}{\partial x_2 \partial x_3} F.$$

For any $F \in S$ we define the ideal:

$$\mathrm{Ann}(F) := \{\Theta \in T \mid \Theta \lrcorner F = 0\}.$$

We always have that $T/\mathrm{Ann}(F)$ is a finite local Gorenstein $\mathbb{C}$-algebra, and all finite local Gorenstein $\mathbb{C}$-algebras are isomorphic to $T/\mathrm{Ann}(F)$ for some $F \in S$. If F is homogenoeus, then $T/\operatorname{Ann}(F)$ is in addition graded, and all finite graded Gorenstein $\mathbb{C}$-algebras are obtained in this way. In the graded case, the Hilbert function of $\operatorname{Ann}(F)$ is symmetric, that is, for $F \in S_d$, we have:

$$\dim\left(T/\operatorname{Ann}(F)\right)_i = \dim\left(T/\operatorname{Ann}(F)\right)_{d-i}$$

(in fact, the two vector spaces $\left(T/\operatorname{Ann}(F)\right)_i$ and $\left(T/\operatorname{Ann}(F)\right)_{d-i}$ are dual to one another).

Proposition 3.1 (Apolarity lemma for linear spans)**.** *Suppose $R \subset \mathbb{P}V$ is a subscheme and $F \in \operatorname{Sym}^d V$. Then*

$$I(R) \subset \operatorname{Ann}(F) \Longleftrightarrow F \in \langle \nu_d(R) \rangle .$$

Proposition 3.2 (Apolarity lemma for Waring rank)**.** *Suppose $F \in \operatorname{Sym}^d V$ and r is an integer. Then*

$$R_{\nu_d(\mathbb{P}(V))}(F) \leq r \Longleftrightarrow \exists \text{ radical homogeneous ideal } I \subset \operatorname{Ann}(F) \text{ with } H_I(t) = r \text{ for all } t \gg 0.$$

4 Identifiability

Fix a nondegenerate subvariety $X \subset \mathbb{P}(V)$ and an integer r.

We say that X is **r-identifiable** if for a general point $p \in \sigma_r(X)$ there is a unique (up to permutation) collection of r points $x_1, \ldots, x_r \in X$ such that $p \in \langle x_1, \ldots, x_r \rangle$. Equivalently, X is r-identifiable if and only if the map $\mathbb{S}_r(X) \to \sigma_r(X)$ is birational.

We say that X is **not r-identifiable** if for a general point $p \in \sigma_r(X)$ there is more than unique collection of r points $x_1, \ldots, x_r \in X$ such that $p \in \langle x_1, \ldots, x_r \rangle$. Note that if $\sigma_r(X)$ is defective, then it is not identifiable.