On the dimension of linear systems with multiple points

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Interpolation problem

How many curves of degree 5 exist in the plane with
1 quartuple point, 1 triple point, 1 double point, and 2 simple points?

Assume that the points are given in general position.

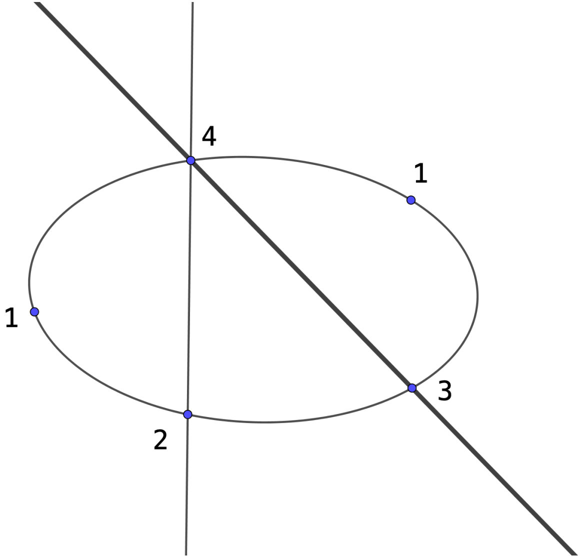
The family of quintic curves has dimension 20.

- A simple point gives 1 condition,
- a double point 3 conditions,
- a triple point 6 conditions,
- a quartuple point 10 conditions.

Hence:

$$20 - 10 - 6 - 3 - 2 = -1$$

We expect that there exists no such curve.


4
1
1
3
2

Special linear systems

Given a linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$, the expected dimension of $\mathcal{L}$ is

$$\mathrm{edim}(\mathcal{L}) = \max\left(\binom{n+d}{n} - \sum_{i=1}^{s}\binom{n+m_i-1}{n} - 1, -1\right)$$

The speciality is exactly the difference between the dimension and the expected dimension:

$$\dim(\mathcal{L}) - \mathrm{edim}(\mathcal{L}) = \mathrm{h}^1(\mathcal{O}(d) \otimes \mathcal{I}_Z) \geq 0$$

where $Z = \{p_1^{m_1} \cup \ldots \cup p_s^{m_s}\}$

$$\mathcal{L}_{n,d}(m_1, \ldots, m_s) \Leftrightarrow \left\{D \in \mathrm{Pic}(\mathrm{Bl}_s(\mathbb{P}^n)) : D = dH - \sum m_i E_i\right\}$$

Section 1

Homogeneous linear systems with double points

Alexander-Hirschowitz Theorem (1995)

A linear system $\mathcal{L}_{n,d}(2^s)$ is special only in the following cases:

- $d = 2,\ 2 \leq s \leq n$
- $d = 3,\ n = 4,\ s = 7$
- $d = 4,\ (n, s) = (2, 5), (3, 9), (4, 14)$

A linear system $\mathcal{L}_{n,d}(2^s)$ is special.

$$\Updownarrow$$

The s-secant variety to the Veronese variety $\nu_d(\mathbb{P}^n)$ is defective.

$$X \subset \mathbb{P}^N \text{ is } s\text{-defective if } \dim \sigma_s(X) < \min(N, s\dim(X) + s - 1).$$

Secant varieties and Waring rank

The Waring rank is the smallest integer r such that a given form (homogeneous polynomial) $f \in S^d\mathbb{C}^{n+1}$ can be written as a sum of r d-th powers of linear forms.

Alexander-Hirschowitz Theorem

A general form in $S^d\mathbb{C}^{n+1}$ has Waring rank $\lceil \frac{\binom{n+d}{n}}{n+1} \rceil$ except if:

- $d = 2, n \geq 2$, where the generic rank is $n + 1$
- $S^3\mathbb{C}^5$, where the generic rank is 8
- $S^4\mathbb{C}^{n+1}, n = 2, 3, 4$, where the generic rank is $\binom{n+2}{2}$

Waring rank of a form $\Leftrightarrow$ symm. rank of a symm. tensor

Segre Varieties

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_k) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_k)$$

Matrices ($k = 2$) are always defective:

$$\dim(\sigma_s(\mathbb{P}^n \times \mathbb{P}^m)) = s(m + n + 1) - 1 - s(s - 1)$$

$$\mathrm{expdim}(\sigma_s(\mathbb{P}^n \times \mathbb{P}^m)) = s(m + n + 1) - 1$$

Defective cases for $k \geq 3$:

- unbalanced cases: $n_k \geq \prod_1^{k-1}(n_i + 1) - \sum_1^{k-1}(n_i + 1) + 1$
- $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3, \quad s = 5$
- $\mathbb{P}^2 \times \mathbb{P}^{2n} \times \mathbb{P}^{2n}, \quad s = 3n + 1$
- $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n, \quad s = 2n + 1$

(Catalisano-Geramita-Gimigliano, Abo-Ottaviani-Peterson)

Conjecture (AOP, 2009)

Let $k \geq 3$. Every Segre variety of the form $(\mathbb{P}^n)^k$ is not defective, except for $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

True for $k = 3$, Strassen-Lickteig.

Conjecture (AOP, 2009)

In the case of arbitrary dimensions, there are no other defective cases except the previous list.

Know for: low dimensions, $s \leq 6$, asymptotically,. . .

Segre-Veronese Varieties: $(\mathbb{P}^{n_1} \times \dots \mathbb{P}^{n_k}, \mathcal{O}(d_1, \dots, d_k))$

$$\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_k) \overset{(\nu_{d_1}, \dots, \nu_{d_k})}{\hookrightarrow} \mathbb{P}(S^{d_1} V_1 \otimes \dots \otimes S^{d_k} V_k)$$

Partially symmetric tensors = Multi-homogeneous polynomials.

Rank $\leq$ Partially symmetric rank $\leq$ Symmetric rank

Notion of X-rank for Segre, Segre-Veronese, Veronese varieties.

Simultaneous rank

Given $X \subseteq \mathbb{P}^N$ and a set of points $\mathcal{F} \subseteq \mathbb{P}^N$. The simultaneous X-rank of $\mathcal{F}$, denoted $R_X(\mathcal{F})$, is the minimal number of points on X whose linear span contains $\mathcal{F}$.

$$R_X(\{q_1, \dots, q_{r+1}\}) = R_{\mathbb{P}^r \times X}(p)$$

Question (Sturmfels)

Two general symmetric $n \times n$ matrices can be simultaneously diagonalized. This implies that two general quadrics in n variables have a simultaneous Waring decomposition of rank n. What is the simultaneous Waring rank for three or more general quadrics?

$\Rightarrow$ We need to study the Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1,2))$

Segre-Veronese of bidegree $(1, 2)$

Conjecture (Abo-B. 2009)

A Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1, 2))$ is s-defective if and only if:

- $\binom{n+2}{2} - n < s < \min\{m + 1, \binom{n+2}{2}\}$ (unbalanced)
- $(m, n) = (2, 2k + 1)$, $s = 3k + 2$ for any $k \geq 1$
- $(m, n) = (4, 3)$, $s = 6$

Exceptions studied by Catalisano-Geramita-Gimigliano, Carlini-Chipalkatti, Ottaviani.

The conjecture is true for: $m = 1$ (Carlini-Chipalkatti), $m = 2$ (Abo-B), $m = 3$ (Galuppi-Oneto).

Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1,2))$: more details

- We may assume balanced: $m \leq \binom{n+2}{2} - n$.
- AB 2009: Specialization on a codimension 2 subspace.

  Subabundant case: bound $\underline{s}(m,n)$. Sharp if $n \geq m^3$.

  Superabundant case: bound $\overline{s}(m,n)$.
- Abo 2010: $(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}(1,2))$ and $(\mathbb{P}^{n+1} \times \mathbb{P}^n, \mathcal{O}(1,2))$ satisfy the conjecture.

Segre-Veronese of bidegree $(1, d)$ (simultaneous Waring rank)

Conjecture

A Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1, d))$ (with $d \geq 3$) is s-defective if and only if:

- $\binom{n+d}{d} - n < s < \min\{m + 1, \binom{n+d}{d}\}$ (unbalanced)
- $(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1, 3))$, $s = 5$

Studied by Bernardi, Carlini, Catalisano, Geramita, Gimigliano, ...

Segre-Veronese of bidegree $(2, d)$

Conjecture

A Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(2, d))$ (with $d \geq 2$) is defective if and only if it belongs to this list:

- $(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}(2, 2))$
- $(\mathbb{P}^2 \times \mathbb{P}^n, \mathcal{O}(2, 2))$
- $(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(2, 2))$
- $(\mathbb{P}^3 \times \mathbb{P}^4, \mathcal{O}(2, 2))$
- $(\mathbb{P}^m \times \mathbb{P}^1, \mathcal{O}(2, 2k))$

Studied by Catalisano, Geramita, Gimigliano, Carlini, Bocci, Abrescia,. . .

Segre-Veronese of bidegree (a, b) with $a, b \geq 3$

Theorem

A Segre-Veronese $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(a, b))$ with $a \geq 3$ and $b \geq 3$ is never defective.

Proof:

- (AB 2013) Inductive proof: Horace method.
- (Galuppi-Oneto 2022) Initial steps: bidegree $(3, 3)$, $(3, 4)$, $(4, 4)$. Collision of fat points.

Section 2

Linear systems with multiple points in $\mathbb{P}^n$

$$\mathcal{L}_{n,d}(m_1, \ldots, m_s)$$

The Seg-Har-Gim-Hir Conjecture

An irreducible curve in $\mathbb{P}^2$ blown up at s general points is a (-1)-curve if it has arithmetical genus 0 and self-intersection -1.

SHGH-Conjecture (1961-86-87-89)

A linear system in $\mathbb{P}^2$ blown up at s general points is special if and only if it is (-1)-special, that is if there is a multiple (-1)-curve in its base locus.

The conjecture is proved:

- up to 9 points (Castelnuovo 1891)
- up to quartuple points (Mignon 1998)
- up to multiplicity 12 for homogeneous systems (Ciliberto-Miranda 1998)
- up to multiplicity 6 for quasi-homogeneous systems (Ciliberto-Miranda 1998, Seibert and Laface 1999, Laface-Ugaglia 2003, Kunte 2005)
- up to points of multiplicity 11 (Dumnicki-Jarnicki 2007)
- up to multiplicity 42 for homogeneous systems (Dumnicki 2007)
- for homogeneous systems with $s \geq 4m^2$ (Roé 2014)
- ...

Nagata conjecture

Nagata Conjecture (1960)

If $s \geq 9$, if a linear system $\mathcal{L}_{2,d}(m_1, \ldots, m_s)$ is not empty, then

$$\sqrt{s}d \geq m_1 + \ldots + m_s$$

and strict inequality holds for $s \geq 10$.

True if s is a square.

See e.g. Ciliberto, Harbourne, Miranda, Roe (2013).

Higher dimension: base locus and speciality

Examples

- $\mathcal{L}_{2,4}(3, 2^2, 1)$: $\dim = \text{edim} = 1$, two simple lines
- $\mathcal{L}_{2,5}(4, 3, 2, 1^2)$: $\dim = 0$, $\text{edim} = -1$, a double line
- $\mathcal{L}_{3,4}(2^8)$: $\dim = \text{edim} = 1$, a double elliptic quartic curve
- $\mathcal{L}_{3,6}(3^8, 2)$: $\dim = 1$, $\text{edim} = -1$, a double quadric surface
- $\mathcal{L}_{4,5}(3^6, 4)$: $\dim = 7$, $\text{edim} = 0$, a double rational normal curve and six double lines

Obstructions: Some (multiple) subvarieties in the base locus of $\mathcal{L}$ give speciality.

General questions

Problem 1: understand the obstruction.

Find the subvarieties which give speciality whenever contained (with some multiplicity) in the base locus of a linear system.

Problem 2: compute the dimension.

Give a formula for the dimension, computing the contribution of each obstruction subvariety.

A nice family of examples: Mori Dream Spaces

Hu-Keel introduced the notion of Mori Dream Spaces:

Definition

A normal projective $\mathbb{Q}$-factorial variety X is called a Mori dream space if the following conditions hold:

1. $\mathrm{Pic}(X)$ is finitely generated, that is $h^1(X, \mathcal{O}_X) = 0$,
2. $\mathrm{Nef}(X)$ is generated by the classes of finitely many semi-ample divisors,
3. there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : S \dashrightarrow X_i$, such that each X_i satisfies (2), and

$$\mathrm{Mov}(X) = \cup f_i^*(\mathrm{Nef}(X_i)).$$

$\mathbb{P}^n$ blown up at points and Mori Dream Spaces

Theorem (Mukai, Castravet-Tevelev)

The variety $\mathrm{Bl}_s(\mathbb{P}^n)$ ($\mathbb{P}^n$ blown up at s general points) is a Mori Dream Space if and only it is in the following list:

- $n = 2$ and $s \leq 8$,
- $n = 3$ and $s \leq 7$,
- $n = 4$ and $s \leq 8$,
- $n \geq 5$ and $s \leq n + 3$.

Answers in the cases of Mori Dream Spaces

Classification of the Stable Base Locus Subvarieties

- $\mathbb{P}^n$ and $s \leq n+2$: linear subspaces (BDP 2015),
- $\mathbb{P}^n$ and $s = n+3$: linear subspaces, rational normal curves, their secant varieties and joins (BDP 2016 + Mukai 2005),
- $\mathbb{P}^3$ and $s = 7$ and $\mathbb{P}^4$ and $s = 8$: Weyl cycles (BDP 2022 + Casagrande-Codogni-Fanelli 2019)

Problems 1 and 2

- $\mathbb{P}^n$ and $s \leq n+2$: the formula is proved (BDP 2015).
- $\mathbb{P}^n$ and $s = n+3$: the formula is conjectured in BDP 2016 and proved in Laface-Postinghel-Santana Sanchez 2022.
- $\mathbb{P}^3$ and $s = 7$ and $\mathbb{P}^4$ and $s = 8$: the formula is conjectured (BDP 2022). Proof in progress.

Linear obstructions and linear speciality

- L_I the linear subspace spanned by p_i, $i \in I \subseteq \{1, \ldots, s\}$,
- $k_I = \sum m_i - (|I| - 1)d$ the multiplicity in the base locus
- $(-1)^{|I|}\binom{n-|I|+k_I}{n}$ the contribution to speciality

The linear expected dimension of $\mathcal{L}$, is

$$\mathrm{ldim}(\mathcal{L}) = \sum_I (-1)^{|I|}\binom{n-|I|+k_I}{n} - 1$$

We say that a linear system $\mathcal{L}$ is linearly special if

$$\dim(\mathcal{L}) \neq \mathrm{ldim}(\mathcal{L})$$

Theorem (B-Dumitrescu-Postinghel 2015)

Any linear system in $\mathbb{P}^n$ with at most $n+2$ base points is linearly non-special.

Fröberg-Iarrobino conjectures

The weak Fröberg-Iarrobino conjecture states that, for any homogeneous linear system $\mathcal{L}$,

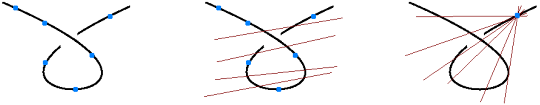
$$\mathrm{ldim}(\mathcal{L}) \leq \dim(\mathcal{L})$$

The strong Fröberg-Iarrobino conjecture is a classification of all the linearly special homogeneous systems:

- $n+3$ or $n+4$ points in $\mathbb{P}^n$
- 7 or 8 points in $\mathbb{P}^2$
- 9 points in $\mathbb{P}^3$
- 14 points in $\mathbb{P}^4$

Linear systems with at least $n + 3$ points

Non-linear obstructions appear:



- C the rational normal curve: $k_C = \sum_1^{n+3} m_i - nd$
- σ_t the t-secant variety to C: $k_{\sigma_t} = tk_C - (t-1)d$
- $J(L_I, \sigma_t)$ the join: $k_{I,\sigma_t} = \sum_{i \in I} m_i + tk_C - (|I| + t - 1)d$

Secant linear expected dimension

The dimension of the join J_{I,σ_t} is

$$r_{I,\sigma_t} = |I| + 2t - 1$$

The contribution of the join J_{I,σ_t} is

$$(-1)^{|I|}\binom{n + k_{I,\sigma_t} - r_{I,\sigma_t} - 1}{n}$$

The secant linear expected dimension of $\mathcal{L}$ is

$$\sigma\mathrm{ldim}(\mathcal{L}) = \sum_I (-1)^{|I|}\binom{n + k_{I,\sigma_t} - r_{I,\sigma_t} - 1}{n} - 1$$

Example

The linear system $\mathcal{L} = \mathcal{L}_{6,8}(6^9)$ is linearly special:

$$\mathrm{ldim}(\mathcal{L}) = -148, \qquad \dim(\mathcal{L}) = 0, \qquad \mathcal{L} = \{2\sigma_3(C)\}$$

The singular locus contains

- the rational normal curve C through the 9 points, $k_C = 6$,
- the 9 cones $J(p_i, C)$ (surfaces) for any p_i, $k_{J(p_i,C)} = 4$,
- the secant variety $\sigma_2(C)$ (threefold), $k_{\sigma_2} = 4$.

The dimension computation is

$$\dim(\mathcal{L}) = \mathrm{ldim}(\mathcal{L}) + \binom{6+6-1-1}{6} - 9 \cdot \binom{6+4-2-1}{6} + \binom{6+4-3-1}{6}$$

The case of $\mathbb{P}^3$

A linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ is *Cremona reduced* if $\sum_{i=1}^{s} m_i - (n-1)d \leq 0$.

Laface-Ugaglia Conjecture (2006)

A Cremona reduced special linear system in $\mathbb{P}^3$ is

- either linearly non-special (and in particular contains multiple lines in the base locus),
- or it contains a quadric surface in the base locus.

- up to 8 points in $\mathbb{P}^3$ (De Volder-Laface 2007)
- up to quintuple points in $\mathbb{P}^3$ (Ballico-B-Caruso-Sala 2012)
- for homogeneous linear systems $\mathcal{L}_{3,d}(m^9)$ of degree $d \leq 2m+1$ in $\mathbb{P}^3$ (B-Dumitrescu-Postinghel 2016)
- up to multiplicity 8 for 9 points in $\mathbb{P}^3$ (BDP 2016)

Quadric hypersurfaces

The speciality given by quadric hypersurfaces is less clear than the speciality given by linear cycles.

- It is sporadic: we expect that a quadric hypersurface produces speciality only in $\mathbb{P}^3$ and in $\mathbb{P}^4$.
- A quadric surface in $\mathbb{P}^3$ can give speciality even if it is simple in the base locus. E.g., for $\mathcal{L} = \mathcal{L}_{3,8}(4^7, 3^2)$, we have $\dim(\mathcal{L}) = 5 = \mathrm{edim}(\mathcal{L}) + 1$.
- The computation of the speciality is difficult in general. It seems impossible to produce a nice formula for the *expected dimension* which takes in account also the quadric contribution.

General techniques

- Horace method (Alexander-Hirschowitz). Specialization of points on a hyperplane, double induction on degree and dimension.
- Specialization on higher codimension subspace (Chandler cod. 2, B-Ottaviani cod. 3, combinatorical generalization by Torrance-Vannieuwenhoven).
- Degeneration of the spaces (Ciliberto-Miranda and their school)
- Toric degeneration, combinatorical techniques (Brannetti, Dumitrescu, Baur-Draisma, Laface-Massarenti-Rischter)
- Collision of fat points (Evain, Mella, Galuppi, Oneto).
- Computational approach (Dumnicki,. . .)

Thank you!