Robust Eigenvectors of Symmetric Tensors

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Eigenvectors of symmetric tensors

Definition

Let $\mathcal{T} \in S^d(\mathbb{R}^n)$. A vector $\mathbf{v} \in \mathbb{C}^n$ is an *eigenvector* of $\mathcal{T}$ with *eigenvalue* $\mu \in \mathbb{C}$ if

$$\mathcal{T} \cdot \mathbf{v}^{d-1} = \mu \mathbf{v}$$

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Definition (De Lathauwer, 1995)

The eigenvectors of $\mathcal{T}$ are the fixed points (up to sign) of an iterative method called the *tensor power method*, given by

$$\mathbf{x}_k \mapsto \mathbf{x}_{k+1} = \frac{\mathcal{T} \cdot \mathbf{x}_k^{d-1}}{\|\mathcal{T} \cdot \mathbf{x}_k^{d-1}\|}$$

We call the vector $\mathbf{x}_0$ an *initializing vector*.

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- When $d = 2$, this is the well-known *matrix power method*

The tensor power method

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Definition

A *robust eigenvector* of $\mathcal{T} \in S^d(\mathbb{R}^n)$ is an eigenvector $\mathbf{v} \in \mathbb{R}^n$ that is an attracting fixed point of the tensor power method, i.e. there exists an $\epsilon > 0$ such that the tensor power method converges to $\mathbf{v}$ for all initializing vectors $\mathbf{x}_0 \in B_\epsilon(\mathbf{v})$ in the ball of radius ϵ centered at $\mathbf{v}$.

Points of caution

- A tensor $\mathcal{T}$ with symmetric decomposition (not necessarily minimal)

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- The only robust eigenvector of a generic symmetric matrix is the one whose eigenvalue is largest in absolute value
- Unlike symmetric matrices, symmetric tensors can have several robust eigenvectors

Why eigenvectors of symmetric tensors?

- If $\mathcal{T}$ has symmetric decomposition (not necessarily minimal)

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- The eigenvectors of $\mathcal{T}$ are the critical points of the symmetric best rank-one approximation problem:

$$\min_{c \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^n} \|\mathcal{T} - c\mathbf{v}^{\otimes d}\|$$

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- We therefore want sufficient conditions for when an eigenvector $\mathbf{v}_i$ is robust so that we can reliably recover it from the tensor power method

Main theorem

Theorem (M-Robeva-Usevich)

For $d \in \mathbb{N}$, let $\mathcal{T}_d \in S^d(\mathbb{R}^n)$ be a tensor with symmetric (not necessarily minimal) decomposition

$$\mathcal{T}_d = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}$$

with $\|\mathbf{v}_i\| = 1$ for all i. Then there exists a $D \in \mathbb{N}$ such that for all $d \geq D$, if $\mathbf{v}_j$ is an eigenvector of $\mathcal{T}_d$ with a non-zero eigenvalue, then $\mathbf{v}_j$ is a robust eigenvector of $\mathcal{T}_d$.

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Lemma

Let $\mathbf{x}_* \in \mathbb{R}^n$ be a fixed point of a C^1 function $\phi : U \to \mathbb{R}^n$, and let $\mathbf{J} : U \to \mathbb{R}^{n \times n}$ be its Jacobian matrix. Then $\mathbf{x}_*$ is an attracting fixed point of the iterative method $\mathbf{x}_{k+1} = \phi(\mathbf{x}_k)$ if $\rho(\mathbf{J}(\mathbf{x}_*)) < 1$. Furthermore, if $\rho(\mathbf{J}(\mathbf{x}_*)) > 0$, then the rate of convergence to $\mathbf{x}_*$ is linear.

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- Apply the Lemma to the tensor power method iteration map

$$\phi(\mathbf{x}) = \frac{\mathcal{T}_d \cdot \mathbf{x}^{d-1}}{\|\mathcal{T}_d \cdot \mathbf{x}^{d-1}\|}$$

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- Show that $\mathbf{J}$ is symmetric and bound its spectral radius

Equiangular sets

Definition

An *equiangular set* (ES) is a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_r \in \mathbb{R}^n$ with $r \geq n$ for which there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = |\langle \mathbf{v}_i, \mathbf{v}_j \rangle|, \ \forall i \neq j \qquad \text{and} \qquad \|\mathbf{v}_i\| = 1, \ \forall i$$

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- If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form an ES, and if $\sigma_{i,j} = \text{sgn}(\langle \mathbf{v}_i, \mathbf{v}_j \rangle) \in \{1, -1\}$ for $i \neq j$, then

$$\mathbf{V}^\top \mathbf{V} = \begin{pmatrix} - & \mathbf{v}_1 & - \\ & \vdots & \\ - & \mathbf{v}_r & - \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_r \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{1,2}\alpha & \ldots & \sigma_{1,r-1}\alpha & \sigma_{1,r}\alpha \\ \sigma_{2,1}\alpha & 1 & \ldots & \sigma_{2,r-1}\alpha & \sigma_{2,r}\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{r-1,1}\alpha & \sigma_{r-1,2}\alpha & \ldots & 1 & \sigma_{r-1,r}\alpha \\ \sigma_{r,1}\alpha & \sigma_{r,2}\alpha & \ldots & \sigma_{r,r-1}\alpha & 1 \end{pmatrix}$$

Explicit bounds on D for ES-decomposable tensors

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Theorem (M-Robeva-Usevich)

Let $\mathcal{T}$ be a tensor generated by an ES $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n$ and coefficients $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$:

$$\mathcal{T} = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}$$

If d is odd and $(\lambda_1, \ldots, \lambda_r) \in Ker(\mathbf{V})$, then all of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors of $\mathcal{T}$. Furthermore, all of these eigenvectors are robust if

$$\frac{\|\mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top\|_2 \alpha^{d-2}(d-1)}{\left(\min_{i\in[r]} |\lambda_i|\right)\left(1-\alpha^{d-1}\right)} < 1$$

which always holds when d is sufficiently large.

Equiangular tight frames

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An equiangular set is an *equiangular tight frame* (*ETF*) if, in addition,

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- If $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{R}^n$ are any unit vectors with $r \geq n$, then the *Welch bound* always holds:

$$\max_{\substack{i,j \in [r] \\ i \neq j}} |\langle \mathbf{u}_i, \mathbf{u}_j \rangle| \geq \sqrt{\frac{r-n}{n(r-1)}}$$

with equality if and only if $\mathbf{u}_1, .., \mathbf{u}_r$ is an ETF

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  with equality if and only if $\mathbf{u}_1, .., \mathbf{u}_r$ is an ETF
- Thus, $\alpha = \sqrt{\frac{r-n}{n(r-1)}}$ for ETFs

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- ETFs have applications in signal processing, coding theory, and quantum information processing

Explicit bounds on D for ETF-decomposable tensors

Theorem (M-Robeva-Usevich)

Let $\mathcal{T}$ be a tensor generated by an ETF $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n$:

$$\mathcal{T} = \sum_{i=1}^{r} \mathbf{v}_i^{\otimes d}$$

If d is even, then all of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors of $\mathcal{T}$. Furthermore, all of these eigenvectors are robust if

$$\frac{\frac{r}{n}\alpha^{d-2}(d-1)}{1+\alpha^{d-2}\left(\frac{r}{n}-1\right)} < 1$$

which always holds when d is sufficiently large.

Examples of ETF-decomposable tensors

$$\mathcal{T} = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}$$

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- **Regular *n*-simplex tensors**: The ETF $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in \mathbb{R}^n$ consists of the vertices of a regular n-simplex in $\mathbb{R}^n$, with $\alpha = \frac{1}{n}$ and $\sigma_{i,j} = \operatorname{sgn}(\langle \mathbf{v}_i, \mathbf{v}_j \rangle) = -1$ for all $i \neq j$

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Figure: The Mecedes-Benz frame (regular 2-simplex) in $\mathbb{R}^2$

Robust eigenvectors of regular n-simplex tensors

Theorem (M-Robeva-Usevich)

Let $\mathcal{T} = \sum_{i=1}^{n+1} \mathbf{v}_i^{\otimes d}$ be a tensor generated by a regular n-simplex frame $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in \mathbb{R}^n$. Then all of $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ are robust eigenvectors for $\mathcal{T}$ for $n \geq 2$ and $d \geq 3$ such that $n + d \geq 7$.

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n \ d	2	3	4	5	6	7	8	9	10
2	✗	✗	✗	✓	✓	✓	✓	✓	✓
3	▲	▲	✓	✓	✓	✓	✓	✓	✓
4	▲	✓	✓	✓	✓	✓	✓	✓	✓
5	▲	✓	✓	✓	✓	✓	✓	✓	✓
6	▲	✓	✓	✓	✓	✓	✓	✓	✓
7	▲	✓	✓	✓	✓	✓	✓	✓	✓
8	▲	✓	✓	✓	✓	✓	✓	✓	✓
9	▲	✓	✓	✓	✓	✓	✓	✓	✓
10	▲	✓	✓	✓	✓	✓	✓	✓	✓

Figure: ✓ and ✗ are guaranteed success and failure of convergence, ▲ is no convergence in numerical experiment

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Theorem (M-Robeva-Usevich)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ be vectors of a regular 2-simplex frame. If $\mathbf{x}_0 \in \mathbb{R}^2$ and there is a unique $\mathbf{v} \in \{\mathbf{v}_1, -\mathbf{v}_1, \mathbf{v}_2, -\mathbf{v}_2, \mathbf{v}_3, -\mathbf{v}_3\}$ which maximizes $\langle \mathbf{v}, \mathbf{x}_0 \rangle$, then the tensor power method with initializing vector $\mathbf{x}_0$ applied to the tensor

$$\mathcal{T} = \mathbf{v}_1^{\otimes d} + \mathbf{v}_2^{\otimes d} + \mathbf{v}_3^{\otimes d}$$

will converge to $\mathbf{v}$, for all even $d \geq 6$. If $d = 4$, then any initializing vector $\mathbf{x}_0$ is a fixed point of the tensor power method.

Regions of convergence for regular 2-simplex tensors

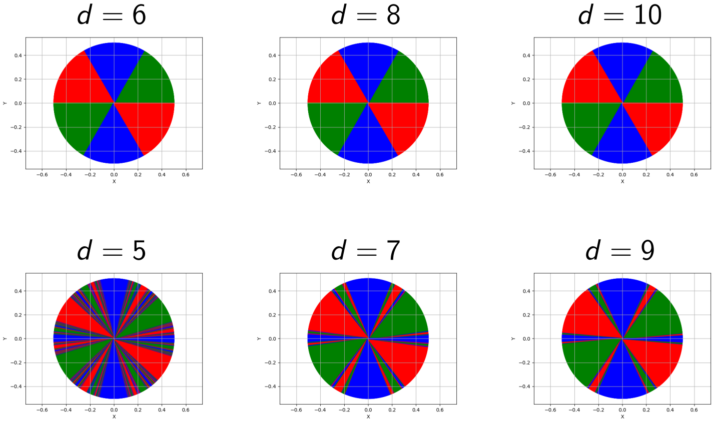

Figure: Regions of convergence on the unit disk in $\mathbb{R}^2$ of the tensor power method for different values of d of the Mercedez-Benz tensor.

Complex eigenvectors of regular n-simplex tensors

d	eigenvector	eigenvalue	multiplicity
10	$(0,1)^\top$	$\frac{513}{512}$	1
	$(\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top$	$\frac{513}{512}$	1
	$(-\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top$	$\frac{513}{512}$	1
	$(-\frac{\sqrt{2}i}{2}, \frac{\sqrt{2}}{2})^\top$	0	2
	$(\frac{\sqrt{2}i}{2}, \frac{\sqrt{2}}{2})^\top$	0	2
	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})^\top$	$\frac{243}{512}$	1
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Table: All eigenvectors of the Mercedes-Benz tensor for $d = 10$

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Conjecture

All eigenvectors of a regular n-simplex tensor are complex, except for the vectors in the frame, and when $d \geq 6$ and even, the vectors on the boundary of the regions of convergence.

More ETF-decomposable tensors

- The diagonal lines of a **cube**:

$$\mathbf{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

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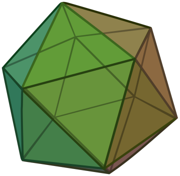
$$\mathbf{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

- The diagonal lines of an **icosahedron**:

$$\mathbf{V} = \frac{1}{\sqrt{1+\varphi^2}} \begin{pmatrix} 0 & 0 & 1 & -1 & \varphi & -\varphi \\ 1 & -1 & \varphi & \varphi & 0 & 0 \\ \varphi & \varphi & 0 & 0 & 1 & 1 \end{pmatrix}$$



More ES-decomposable tensors

- These 16 equiangular lines in $\mathbb{R}^6$ form an ETF:

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- These 6 vectors in $\mathbb{R}^4$ are an ES that is not an ETF:

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{6}}{3} \end{pmatrix}$$

More open problems

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Explain the fractal regions of convergence for the regular 2-simplex tensors when d is odd.

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Conjecture

The eigenvectors of an ETF-decomposable tensor are stable under perturbation, i.e. if a tensor with small Frobenius norm is added, then the eigenvectors of this new tensor can be retrieved by the tensor power method and they will be close to the original eigenvectors.