# Tensors and Equilibria in Game Theory

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Example (Bach or Stravinsky)

Two people would like to go to a classical music concert together. The first person prefers to see a *Bach concert* $= 1$, whereas the second prefers a *Stravinsky concert* $= 2$.

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## Example (Bach or Stravinsky)

Two people would like to go to a classical music concert together. The first person prefers to see a *Bach concert* = 1, whereas the second prefers a *Stravinsky concert* = 2. We consider the following payoff matrices:

| | | Player 2 | |
|---|---|---|---|
| | | Bach | Stravinsky |
| Player 1 | Bach | $(3, 2)$ | $(0, 0)$ |
| | Stravinsky | $(0, 0)$ | $(2, 3)$ |

# Set up for a 2-player game in normal form

Example (2 × 2 game: Bach or Stravinsky)

| | | Player 2: Bach | Player 2: Stravinsky |
|---|---|---|---|
| Player 1 | Bach | $(3,2)$ | $(0,0)$ |
| | Stravinsky | $(0,0)$ | $(2,3)$ |

$$X^{(1)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \ X^{(2)} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

e.g. $X_{21}^{(1)} = 0$ represents the payoff of Player 1 when the Player 1 chooses the strategy 2 (Stravinsky concert) and Player 2 chooses the strategy 1 (Bach concert).

# Set up for a $n$-player game in normal form

- The Player 1 can choose from $d_1$ *pure strategies* and the Player 2 can choose from $d_2$ *pure strategies* etc.

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$$p_j^{(i)} \geqslant 0, \text{ for all } j \in [d_i] \text{ and } \sum_{j=1}^{d_i} p_j^{(i)} = 1. \tag{1}$$

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$$p_j^{(i)} \geqslant 0, \text{ for all } j \in [d_i] \text{ and } \sum_{j=1}^{d_i} p_j^{(i)} = 1. \tag{1}$$

- $(p_1^{(i)}, \ldots, p_{d_i}^{(i)})$ is a point in the probability simplex $\Delta_{d_i - 1}$.
- Each Player $i$ has a $d_1 \times \cdots \times d_n$ *payoff tensor* $X^{(i)}$.

# Expected payoff of a player

The entry $X^{(i)}_{j_1,\ldots j_n} \in \mathbb{R}$ represents the value of the payoff of the Player $i$, when Player 1 selects the pure strategy $j_1$ and the player 2 selects the pure strategy $j_2$ etc. The expected payoff for Player $i$ is:

$$PX^{(i)} := \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\ldots j_n} p^{(1)}_{j_1} \cdots p^{(n)}_{j_n}$$

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## Example (Bach or Stravinsky)

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Player 1's expected payoff $= PX^{(1)} = 3p^{(1)}_1 p^{(2)}_1 + 2p^{(1)}_2 p^{(2)}_2$

Player 2's expected payoff $= PX^{(2)} = 2p^{(1)}_1 p^{(2)}_1 + 3p^{(1)}_2 p^{(2)}_2$

# Definition of Nash equilibria

## Definition (Nash equilibrium)

A point $P \in \Delta_{d_1-1} \times \cdots \times \Delta_{d_n-1}$ is called a *Nash equilibrium* for a $n$-player game $X$, if none of the players can increase their expected payoff $PX^{(i)}$ by changing their strategy while assuming the other players have fixed mixed strategies.

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- By the result of Nash in 1950*, there exists a Nash equilibrium for any finite game.
- In 2002, Sturmfels* explores how one can find Nash equilibria by solving a system of polynomial equations.

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# Nash equilibria and multilinear equations

Example (continued)

A point $(p_1^{(1)}, p_2^{(1)}, p_1^{(2)}, p_2^{(2)}) \in \Delta_1 \times \Delta_1$ is a Nash equilibrium, if and only if for all pairs $(u_1, u_2)$ with $u_1, u_2 \geqslant 0$ and $u_1 + u_2 = 1$ we have the following inequalities:

$$\begin{aligned} PX^{(1)} &= 3p_1^{(1)}p_1^{(2)} + 2p_2^{(1)}p_2^{(2)} \geqslant 3u_1p_1^{(2)} + 2u_2p_2^{(2)} \\ PX^{(2)} &= 2p_1^{(1)}p_1^{(2)} + 3p_2^{(1)}p_2^{(2)} \geqslant 2p_1^{(1)}u_1 + 3p_2^{(1)}u_2 \end{aligned} \tag{2}$$

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Since the right hand side of each inequality is a linear function in $(u_1, u_2)$ and since $(p_1^{(1)}, p_2^{(1)}, p_1^{(2)}, p_2^{(2)}) \in \Delta_1 \times \Delta_1$, we obtain the following algebraic set from (2) where the expressions in the parenthesis are all non-negative.

# Nash equilibria and multilinear equations

Example (continued)

$$p_1^{(1)}\left(P.X^{(1)} - \sum_{j_2=1}^{2} X^{(1)}_{1,j_2} p^{(2)}_{j_2}\right) = p_1^{(1)}\left(3p_1^{(1)}p_1^{(2)} + 2p_2^{(1)}p_2^{(2)} - 3p_1^{(2)}\right)$$

$$p_2^{(1)}\left(P.X^{(1)} - \sum_{j_2=1}^{2} X^{(1)}_{2,j_2} p^{(2)}_{j_2}\right) = p_2^{(1)}\left(3p_1^{(1)}p_1^{(2)} + 2p_2^{(1)}p_2^{(2)} - 2p_2^{(2)}\right)$$

$$p_1^{(2)}\left(P.X^{(2)} - \sum_{j_1=1}^{2} X^{(2)}_{j_1,1} p^{(1)}_{j_1}\right) = p_1^{(2)}\left(2p_1^{(1)}p_1^{(2)} + 3p_2^{(1)}p_2^{(2)} - 2p_1^{(1)}\right)$$

$$p_2^{(2)}\left(P.X^{(2)} - \sum_{j_1=1}^{2} X^{(2)}_{j_1,2} p^{(1)}_{j_1}\right) = p_2^{(2)}\left(2p_1^{(1)}p_1^{(2)} + 3p_2^{(1)}p_2^{(2)} - 3p_2^{(1)}\right)$$

where the expressions in the parenthesis are all non-negative.

# Nash equilibria and multilinear equations

- The Nash equlibria $(p_1^{(1)}, p_2^{(1)}, p_1^{(2)}, p_2^{(2)}) \in \Delta_1 \times \Delta_1$ are $(1,0,1,0), (0,1,0,1)$ and $(\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5})$.

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- Compute Nash equilibria fast: `HomotopyContinuation.jl`.
  Check the example online of a $3 \times 3 \times 3$ game and its computation.

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- Compute Nash equilibria fast: `HomotopyContinuation.jl`.
  Check the example online of a $3 \times 3 \times 3$ game and its computation.

  www.juliahomotopycontinuation.org/examples/nash/
- In the general case, one solves $d_1 + \ldots + d_n$ multilinear equations:

$$p_k^{(i)} \left( PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \ldots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \right)$$

where each parenthesized expression is nonnegative.

# Polytopes and Nash equilibria

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- For totally mixed Nash equilibria (strictly positive probabilites), we consider the parenthesized expressions.
- Eliminating the unknowns $PX^{(i)}$ and setting $p_{d_i}^{(i)} = 1 - \sum_{j=1}^{d_i-1} p_j^{(i)}$, $d_1 + \cdots + d_n - n$, we obtain a system of $d_1 + \cdots + d_n - n$ equations in $d_1 + \cdots + d_n - n$ unknowns for $k = 2, 3, \ldots, d_i$:

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$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} \left( X^{(i)}_{j_1 \ldots k \ldots j_n} - X^{(i)}_{j_1, \ldots 1 \ldots j_n} \right) p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n}$$

- One may use Bernstein–Khovanskii–Kushnirenko (BKK) theorem theorem for the maximal number of totally mixed Nash equilibria of a game.

# Universality of Nash equilibria

- One solves multilinear equations of degree $n-1$ in order to obtain the set of Nash equilibria.

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- (Harsanyi, 1973): For the generic case, i.e. generic payoff matrices, the number of Nash equilibria is finite and odd.

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# Universality of Nash equilibria

- One solves multilinear equations of degree $n - 1$ in order to obtain the set of Nash equilibria.
- (Harsanyi, 1973): For the generic case, i.e. generic payoff matrices, the number of Nash equilibria is finite and odd.
- Including the non-generic case, we obtain a strong bridge between Game Theory and (real) Algebraic Geometry.

**Theorem (Datta*, 2003)**

*Every real algebraic variety is isomorphic to the set of totally mixed Nash equilibria of a 3-player game, and also of an n-player game in which each player has two pure strategies.*

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# Correlated equilibrium

- For Nash equilibrium, there is a causal independence for the strategies of the players.

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# Correlated equilibrium

- For Nash equilibrium, there is a causal independence for the strategies of the players.
- Aumann* introduced a new concept of equilibria which allows dependency for the choices of strategies between players.
- In this concept, one considers a joint probability distribution $P = (p_{j_1 \cdots j_n}) \in \Delta_{d_1 d_2 \cdots d_n - 1}$ over the pure strategies of the players.
- A joint probability distribution (or a tensor) is called a *correlated equilibrium*, if no player can raise their expected payoff by breaking their part of the (agreed) joint distribution while assuming that the other players adhere to their own recommendations.

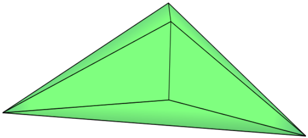
*R. J. Aumann. Subjectivity and correlation in randomized strategies. Journal of Mathematical Economics, 1974

# Correlated equilibrium

- Aumann shows* that this definition is equivalent to the following: A tensor $P \in \Delta_{d_1 \cdots d_n - 1}$ is a *correlated equilibrium* for a game $X$ if and only if

$$\sum_{j_1=1}^{d_1} \cdots \sum_{j_i=1}^{\widehat{d_i}} \cdots \sum_{j_n=1}^{d_n} \left( X^{(i)}_{j_1 \cdots j_{i-1} k j_{i+1} \cdots j_n} - X^{(i)}_{j_1 \cdots j_{i-1} l j_{i+1} \cdots , j_n} \right) p_{j_1 \cdots j_{i-1} k j_{i+1} \cdots j_n} \geqslant 0.$$

for all $k, l \in [d_i]$, and for all $i \in [n]$. The set of all such equilibria is the *correlated equilibrium polytope* $P_G$ of the game $G$.



*R. J. Aumann. Correlated equilibrium as an expression of bayesian rationality, 1987.

# Correlated equilibrium for $2 \times 2$-games

## Example (Bach or Stravinsky)

| | | Player 2 | |
|---|---|---|---|
| | | Bach | Stravinsky |
| Player 1 | Bach | $(3, 2)$ | $(0, 0)$ |
| | Stravinsky | $(0, 0)$ | $(2, 3)$ |

The Nash equilibria are $(1, 0, 0, 0), (0, 0, 0, 1)$ and $(\frac{6}{25}, \frac{9}{25}, \frac{4}{25}, \frac{6}{25})$. A mediator is giving recommendation $(p_{11}, p_{12}, p_{21}, p_{22})$ to the players given their payoff matrices.

$$\begin{aligned}
(X_{11}^{(1)} - X_{21}^{(1)})p_{11} + (X_{12}^{(1)} - X_{22}^{(1)})p_{12} &\geqslant 0 \\
(X_{21}^{(1)} - X_{11}^{(1)})p_{21} + (X_{22}^{(1)} - X_{12}^{(1)})p_{22} &\geqslant 0 \\
(X_{21}^{(2)} - X_{22}^{(2)})p_{21} + (X_{11}^{(2)} - X_{12}^{(2)})p_{11} &\geqslant 0 \\
(X_{22}^{(2)} - X_{21}^{(2)})p_{22} + (X_{12}^{(2)} - X_{11}^{(2)})p_{12} &\geqslant 0
\end{aligned} \qquad (3)$$

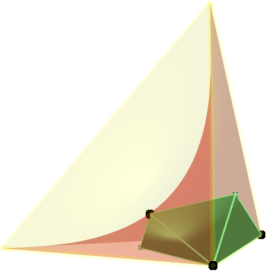
# Correlated equilibrium

## Example (Bach or Stravinsky)

The correlated equilibrium polytope for the Bach or Stravinsky game is

$$C_G = \text{conv}\left((0,0,0,1), \left(\frac{6}{25}, \frac{9}{25}, \frac{4}{25}, \frac{6}{25}\right), (1,0,0,0), \left(\frac{3}{8}, 0, \frac{1}{4}, \frac{3}{8}\right), \left(\frac{2}{7}, \frac{3}{7}, 0, \frac{2}{7}\right)\right)$$

which has 6 facets where Nash equlibria are all vertices of this polytope shown in black. In particular these vertices intersect with the Segre embedding $p_{11}p_{22} - p_{12}p_{21}$ in the probability simplex.



# Combinatorics of correlated equilibrium polytope*

- The correlated equilibrium polytope $P_G$ for $2 \times 2$-games is either a point or a 3-dimensional bipyramid with 5 vertices and 6 facets.

*M.-C. Brandenburg, B. Hollering, and I. Portakal. Combinatorics of correlated equilibria, 2022.

# Combinatorics of correlated equilibrium polytope[*]

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**Theorem (Brandenburg, Hollering, , 2022)**

*Let $G$ be a $(2 \times 3)$-game and $P_G$ be the associated correlated equilibrium polytope. Then one of the following holds:*

- *$P_G$ is a point,*
- *$P_G$ is of maximal dimensional 5 and of a unique combinatorial type,*
- *There exists a $(2 \times 2)$-game $G'$ such that $P_{G'}$ has maximal dimensional 3 is and combinatorially equivalent to $P_G$.*

[*]M.-C. Brandenburg, B. Hollering, and I. Portakal. Combinatorics of correlated equilibria, 2022.

# Combinatorics of correlated equilibrium polytope

| Unique Combinatorial Types by Dimension | | | | | |
|---|---|---|---|---|---|
| Dimension | 0 | 3 | 5 | 7 | 9 |
| $(2 \times 2)$ | 1 | 1 | 0 | 0 | 0 |
| $(2 \times 3)$ | 1 | 1 | 1 | 0 | 0 |
| $(2 \times 4)$ | 1 | 1 | 1 | 3 | 0 |
| $(2 \times 5)$ | 1 | 1 | 1 | 3 | 4 |

Table: The number of unique combinatorial types of $P_G$ of each dimension for a $(2 \times n)$-game in a random sampling of size 100 000.

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| $(2 \times 4)$ | 1 | 1 | 1 | 3 | 0 |
| $(2 \times 5)$ | 1 | 1 | 1 | 3 | 4 |

Table: The number of unique combinatorial types of $P_G$ of each dimension for a $(2 \times n)$-game in a random sampling of size 100 000.

In contrast, $(2 \times 2 \times 2)$-games exhibit a much wider variety of distinct combinatorial types. In a sample of 100 000 random payoff matrices for $(2 \times 2 \times 2)$-games, we found 14 949 distinct combinatorial types which are of maximal dimension. Amongst these 7-dimensional polytopes, the number of vertices can range from 8 to 119, the number of facets from 8 to 14, and the number of total faces from 256 to 2338.

# Dependency equilibrium*

- Recall that for Nash equilibrium, there is a causal independence for the strategies of the players: each player acts independently, without communication and collobaration with the other players.

*W. Spohn. Dependency equilibria and the causal structure of decision and game stituations. 2003.

# Dependency equilibrium*

- Recall that for Nash equilibrium, there is a causal independence for the strategies of the players: each player acts independently, without communication and collobaration with the other players.
- Prisoners' dilemma. C= Cooperate, D = Defect. Both prisoners defecting is the only Nash and correlated equilibrium.

Prisoner 2

| | | C | D |
|---|---|---|---|
| Prisoner 1 | C | $(-1,-1)$ | $(-3,-0)$ |
| | D | $(0,-3)$ | $(-2,-2)$ |

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- In contrast, Spohn introduced in 2003 the concept of *dependency equilibria* where the players simultaneously maximize their *conditional expected payoffs*; appending communication between players.

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# Dependency equilibrium

*Conditional expected payoff* of Player $i$, in case they choose strategy $k \in [d_i]$:

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} \frac{p_{j_1 \cdots k \cdots j_n}}{p_{+ \cdots + k + \cdots +}}. \qquad (4)$$

where $p_{+ \cdots + k + \cdots +}$ is the sum of all probabilities $p_{j_1 j_2 \cdots j_n}$ where $j_i = k$.

# Dependency equilibrium

*Conditional expected payoff* of Player $i$, in case they choose strategy $k \in [d_i]$:

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where $p_{+\cdots+k+\cdots+}$ is the sum of all probabilities $p_{j_1 j_2 \cdots j_n}$ where $j_i = k$.

**Definition (Dependency equilibrium)**

A tensor $P$ in $\Delta := \Delta^{\circ}_{d_1 \cdots d_n - 1}$ is a *dependency equilibrium* for $X$ if the conditional expected payoff of each Player $i$ is independent of their choice $k \in [d_i]$.

This specifies equations among ratios of linear forms:

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} \frac{p_{j_1 \cdots k \cdots j_n}}{p_{+ \cdots + k + \cdots +}}$$

$$= \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k' \cdots j_n} \frac{p_{j_1 \cdots k' \cdots j_n}}{p_{+ \cdots + k' + \cdots +}}$$

We require this for all $i \in [n]$ and all $k, k' \in [d_i]$.

# Determinantal varieties

- By clearing denominators, we obtain quadratic equations in $P$. They define the *Spohn variety* $\mathcal{V}_X \subset \mathbb{P}(V)$. Here $V = \mathbb{C}^{d_1 \times d_2 \times \cdots \times d_n}$.

# Determinantal varieties

- By clearing denominators, we obtain quadratic equations in $P$. They define the *Spohn variety* $\mathcal{V}_X \subset \mathbb{P}(V)$. Here $V = \mathbb{C}^{d_1 \times d_2 \times \cdots \times d_n}$.
- For $i = 1, 2, \ldots, n$, define a matrix with $d_i$ rows and two columns:

$$M_i = \begin{bmatrix} \vdots & \vdots \\ p_{+\cdots+k+\cdots+} & \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k\cdots j_n} p_{j_1\cdots k\cdots j_n} \\ \vdots & \vdots \end{bmatrix}$$

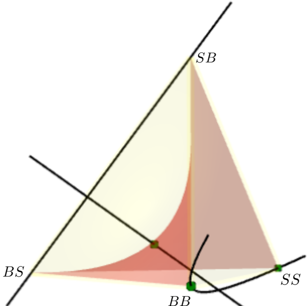
Dependency equilibria for $X$ are tensors $P \in \Delta$ where each $M_i$ has rank one.

# Spohn variety

Example (Bach or Stravinsky)

$$\begin{bmatrix} p_{11}+p_{12} & X^{(1)}_{11}p_{11}+X^{(1)}_{12}p_{12} \\ p_{21}+p_{22} & X^{(1)}_{21}p_{21}+X^{(1)}_{22}p_{22} \end{bmatrix}, \begin{bmatrix} p_{11}+p_{21} & X^{(2)}_{11}p_{11}+X^{(1)}_{21}p_{21} \\ p_{12}+p_{22} & X^{(2)}_{12}p_{12}+X^{(1)}_{22}p_{22} \end{bmatrix}$$

The Spohn variety a quartic curve which is defined by $3p_{11}p_{21}+p_{11}p_{22}-2p_{12}p_{22}$ and $2p_{11}p_{12}-p_{11}p_{22}-3p_{21}p_{22}$ with three irreducible components.



# Motivational quote

- One obtains Nash equilibria by intersecting the dependency equilibria with the Segre variety $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_n}$. or imposing the rank 1 condition i.e.

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$$p_{j_1 \dots j_n} = p^{(1)}_{j_1} \cdots p^{(n)}_{j_n}$$

- Spohn explains the difficulty about dependency equilibria as following:

*The computation of dependency equilibria seems to be a messy business. Obviously it requires one to solve quadratic equations in two-person games, and the more persons, the higher the order of the polynomials we become entangled with. All linear ease is lost. Therefore, I cannot offer a well developed theory of dependency equilibria.*

# Geometry of dependency equilibria*

Theorem (, Sturmfels, 2022)

*If the payoff tables $X$ are generic then the Spohn variety $\mathcal{V}_X$ is irreducible of codimension $d_1 + d_2 + \cdots + d_n - n$ and degree $d_1 d_2 \ldots d_n$. The intersection of $\mathcal{V}_X$ with the Segre variety in the open simplex $\Delta$ is precisely the set of totally mixed Nash equilibria for $X$.*

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**Theorem (, Sturmfels, 2022)**

*If $n = d_1 = d_2 = 2$ then the Spohn variety $\mathcal{V}_X$ is an elliptic curve. In all other cases, the Spohn variety $\mathcal{V}_X$ is rational, represented by a map onto $(\mathbb{P}^1)^n$ with linear fibers.*

---

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# Spohn Conditional Independence Variety

- Drawback of dependency equilibria is that they are abundant: $\mathcal{V}_X \cap \Delta$ of all dependency equilibria has dimension $\prod_{i=1}^{n} d_i - \sum_{j=1}^{n} d_j + n - 1$.

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- Hence, we restrict to intersections of $\mathcal{V}_X$ with statistical models in $\Delta$. Natural candidates are *the conditional independence models*.
- Each CI statement translates into a system of homogeneous quadratic constraints in the entries $p_{j_1 j_2 \cdots j_n}$. We denote the projective variety in $\mathbb{P}(V)$ defined by these quadrics arising from all statements in $\mathcal{C}$ by $\mathcal{M}_{\mathcal{C}}$.

# Spohn Conditional Independence Variety

- Suppose $X$ is any game in normal form, and $\mathcal{C}$ is any collection of CI statements. We define the *Spohn CI variety* to be the intersection of the Spohn variety with the CI model:

$$\mathcal{V}_{X,\mathcal{C}} = \mathcal{V}_X \cap \mathcal{M}_{\mathcal{C}}. \tag{5}$$

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- We focus on the case where all random variables are binary, i.e. $d_1 = d_2 = \cdots = d_n = 2$.

# Dependencies between the strategies of the players

Example (Nash points)

- Let $\mathcal{C}$ be the set of all CI statements on $[n]$: $\mathcal{M}_{\mathcal{C}} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ and the Spohn CI variety is the set of all Nash points in the Spohn variety $\mathcal{V}_X$. This variety is finite, and its cardinality is the number of derangements of $[n]$, which is $1, 2, 9, 44, 265, \ldots$ for $n = 1, 2, 3, 4, 5, \ldots$.

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Question ( mostly open )

*What happens in between?*

# Nash conditional independence curve*

### Conjecture (, Sturmfels, 2022)

*For every conditional independence models* $\mathcal{C}$ *on* $n$ *binary random variables, the Spohn CI variety* $\mathcal{V}_{X,\mathcal{C}}$ *has the expected codimension* $n$ *inside the model* $\mathcal{M}_{\mathcal{C}}$ *in* $\mathbb{P}^{2^n-1}$*. The variety* $\mathcal{V}_{X,\mathcal{C}}$ *is positive-dimensional and irreducible whenever the network has at least one edge.*

* I. Portakal and J. Sendra-Arraz. Nash conditional independence curve, 2022.

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(, Sendra-Arranz, 2022): We prove the conjecture for one edge-models where we call the Spohn CI variety *Nash CI curve*. Moreover we prove a similar universality theorem as Datta's.

### Theorem (, Sendra-Arranz, 2022)

*Let $S \subseteq \mathbb{R}^n$ be a real affine algebraic variety defined by $m$ polynomials with $m < n$. Then, there exists a $N$-person game with binary choices such that an affine open subset of the Spohn CI variety for the one-edge model is isomorphic to $S$.*

*I. Portakal and J. Sendra-Arraz. Nash conditional independence curve, 2022.

Thank you for your time!

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