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On the regularity of natural apolar schemes

Tensors in statistics, optimization and machine learning - AGATES 2022, Warsaw.

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1. A concrete problem: GADs
2. Schemes evincing GADs
3. Regularity: examples
4. Regularity: results
5. Conjectures & Open problems

1 Symmetric tensors

Let V be a $\Bbbk$-vector space of dimension n. The order-d *tensors* are the elements of the tensor space $V^{\otimes d}$. Given a $\Bbbk$-basis $e_1, \dots, e_n$ of V, every tensor $T \in V^{\otimes d}$ may be written as

$$T = \sum_{1 \leqslant i_j \leqslant n} T_{i_1,\dots,i_d} e_{i_1} \otimes \cdots \otimes e_{i_d}.$$

A tensor $T \in V^{\otimes d}$ is *symmetric* if, for every permutation $(i_{\sigma_1}, \dots, i_{\sigma_d})$ of $(i_1, \dots, i_d)$, we have

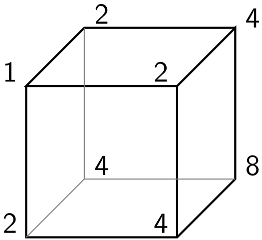
$$T_{i_1,\dots,i_d} = T_{i_{\sigma_1},\dots,i_{\sigma_d}}.$$

1 Symmetric tensors

Symmetric tensors ≡ homogeneous polynomials

$$V^{\otimes d} \longleftrightarrow \Bbbk[X_1, \ldots, X_n]_d,$$

$$\sum_{1 \leqslant i_j \leqslant n} T_{i_1,\ldots,i_d} e_{i_1} \otimes \cdots \otimes e_{i_d} \longleftrightarrow \sum_{1 \leqslant i_j \leqslant n} c(i_j) \cdot T_{i_1,\ldots,i_d} X_{i_1} \cdots X_{i_d}.$$

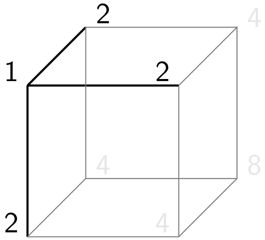


$$\longleftrightarrow \quad x^3 + 6x^2y + 12xy^2 + 8y^3$$

1 The problem

Goal

Finding "good" decompositions of a given symmetric tensor.



$$\longleftrightarrow \qquad (x+2y)^3$$

1 Generalized additive decompositions (GADs)

$$F = 4x^3+12x^2y+6x^2z+18xy^2+6xyz+6xz^2+10y^3+3y^2z+3yz^2+2z^3.$$

GADs of F: Examples & Non-examples

$$= (x+y)^3 + (x+z)^3 + (x+y+z)^3 + (x+2y)^3,$$

$$= (2x+3y+z)(x^2+3xy+xz+3y^2+z^2)$$
$$+ (2x+y+z)(x^2+xy+xz+y^2-yz+z^2),$$

$$= (2x+2y+z)(2x^2+4xy+2xz+5y^2-yz+2z^2),$$

$$= (2x+2y+z)(2x^2+4xy+2xz) + (2x+2y+z)(5y^2-yz+2z^2),$$

$$= (x+y)(x^2+2xy+y^2) + (x+z)^3 + (x+y+z)^3 + (x+2y)^3.$$

1 Generalized additive decompositions (GADs)

Let $\mathcal{S} = \Bbbk[X_0, \dots, X_n] = \oplus_{d \geqslant 0} \mathcal{S}_d$ be the graded polynomial ring.

GAD

Let $F \in \mathcal{S}_d$ and $L_1, \dots, L_s \in \mathcal{S}_1$ be non-proportional linear forms. A *generalized additive decomposition* (GAD) of F *supported at* $(L_1, \dots, L_s)$, is an expression

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i, \qquad \text{where } 0 \leqslant k_i \leqslant d, \text{ for all } i \in \{1, \dots, s\},$$

where $L_i \nmid G_i$, for each $i \in \{1, \dots, s\}$.

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2 Natural apolar schemes

Let $\mathcal{R} = \Bbbk[Y_0, \dots, Y_n] = \oplus_{d \geqslant 0} \mathcal{R}_d$ be the standard dual graded polynomial ring, acting on $\mathcal{S}$ by extending

$$Y_i \circ F = \partial / \partial_{X_i} F$$

Natural apolar scheme

Let $F = L^{d-k} G \in \mathcal{S}_d$ a local GAD.

i. Let $f \in \underline{\mathcal{S}}$ be the dehomogenization of F_{dp} by L.

ii. Compute the ideal

$$f^{\perp} = \{ g \in \underline{\mathcal{R}} \ : \ g \circ f = 0 \}.$$

iii. Let J be the homogenization of $f^{\perp}$ w.r.t. L.

The scheme $Z(J)$ is called the *natural apolar scheme* to F w.r.t. L.

A. Bernardi, J. Jelisiejew, P. M. Marques, K. Ranestad, *On polynomials with given Hilbert function and applications*, Collect. Math. 69 (2018), pp. 39–64.

2 Natural apolar schemes

Effective computation

i. Base-change $(X_0 \longleftarrow L)$: $F' \longleftarrow F$.

ii. Compute the e-truncated Hankel matrix of F' w.r.t. the monomial bases of $\underline{\mathcal{S}}$ and $\underline{\mathcal{R}}$:

$$H^e_{F'} = \left(Y_0^{d-|\alpha+\beta|} Y^{\alpha+\beta} \circ F' \right)_{\substack{\alpha,\beta\in\mathbb{N}^n \\ |\alpha|,|\beta|\leqslant e}} .$$

iii. Compute its kernel as an ideal in $\underline{\mathcal{R}}$:

$$J' = \ker H^{d+1}_{F'} \subset \underline{\mathcal{R}}.$$

iv. Inverse transposed base-change of (i.): $J \longleftarrow J'$.

v. Return the homogenization of J in $\mathcal{R}$.

2 Scheme evincing GADs

Scheme evincing a GAD

Let

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i$$

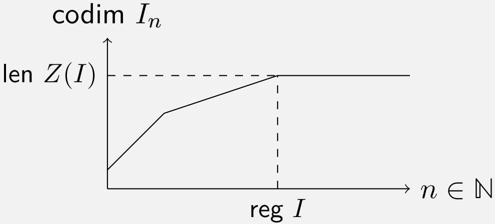
be a GAD of F, and let $Z(J_i)$ be the natural apolar scheme to $L_i^{d-k_i} G_i$ w.r.t. L_i. Then we say that this GAD is *evinced* by

$$Z = \cup_{i=1}^{s} Z(J_i) = Z(\cap_{i=1}^{s} J_i).$$

2 Scheme evincing GADs

Properties

- Z is apolar to F, i.e. $I(Z) \subseteq F^{\perp}$.
- $I(Z)$ is homogeneous and saturated.
- Z is zero(-affine)-dimensional, i.e. it is a scheme of points.
- Therefore, its Hilbert function has this shape:



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3 Back to our example

Same F as before

$$\begin{aligned} L_1^3 + L_2^3 + L_3^3 + L_4^3 &\rightarrow [1, 3, 4, 4, 4, \dots], \quad \star \\ L_5 Q_5 + L_6 Q_6 &\rightarrow [1, 3, 5, 6, 6, \dots], \\ L_7 Q_7 &\rightarrow [1, 3, 4, 4, 4, \dots]. \quad \star \end{aligned}$$

They are all regular in degree $3 = \deg F$.

Q: Is it always the case (at least for the minimal ones)?

On the regularity of *every* non-redundant scheme

Non-redundant schemes

A scheme Z apolar to F is called **non-redundant** if there are no proper subschemes $Z' \subsetneq Z$ apolar to F.

Notice: cactus implies non-redundancy.

Claim

Every non-redundant scheme Z apolar to $F \in \mathcal{S}_d$ is regular in degree d.

Idea of the proof: given $I \subseteq F^\perp$, we produce $I \subseteq J \subseteq F^\perp$ that evinces a GAD of F.

On the regularity of *every* non-redundant scheme

Non-redundant s[illegible]s

A scheme Z apolar t[illegible]called **non-redundant** if th[illegible]no proper subschemes $Z' \subsetneq Z$ apol[illegible]

Notice: cactus implies non-redu[illegible]

Claim

Every non-redundant scheme Z [illegible] to F [illegible] regular in degree d.

Idea of the proof: given [illegible], we produce $I \subseteq J$ [illegible]hat evinces a GAD of F.

3 (Counter)example: irredundant but d-irregular

Let $\mathcal{S} = \mathbb{C}[x, y, z]$ and consider the scheme Z evincing the GAD

$$F = xG_1 + yG_2 \in \mathcal{S}_4,$$

where

$$G_1 = 10x^3 - 4x^2y + 4x^2z - 4xy^2 - 8xyz - 3xz^2 - 8y^3 - 4z^3 \in \mathcal{S}_3,$$
$$G_2 = 5x^3 + 9xy^2 - 5y^3 - 7y^2z + 6yz^2 - z^3 \in \mathcal{S}_3.$$

One can show that

- Z is irredundant, and
- its Hilbert series is $[1, 3, 6, 10, 11, 12, 12, \dots]$, hence Z is not regular in degree $4 = \deg F$.

Conclusion: irredundant schemes may be d-irregular.

Remark: Z is minimal by inclusion, but not by length.

3 (Counter)example: minimal but not evincing a GAD

Let $\mathcal{S} = \mathbb{C}[x, y, z, u, v, w]$ and consider the polynomial

$$F = 60z^3 + 60z^2u + 10y^2x + 70z^2x + 360zux + 60u^2x + 120zvx + 70yx^2 + 75zx^2 + 70ux^2 + 180vx^2 + 10wx^2 + 24x^3 \in \mathcal{S}_3,$$

and the ideal

$$J = \langle w^2, vw, uw, zw, yw, v^2, uv, zv - 6xw, yv, u^2 - 6xw, zu - xv, yu, z^2 - xu, yz, y^2 - xw\rangle \subseteq \mathcal{R}.$$

One can show that

- J is a zero-dim., hom., saturated ideal apolar to F, and
- its Hilbert series is $[1, 6, 6, \dots]$, and Z is the unique cactus scheme for F.

Conclusion: cactus schemes are not always evinced by GADs of F.

Remark: Z is regular in degree 1.

4 Outline

4 Why GADs

Understanding minimal apolar schemes (cactus)

Let Z be a zero-dimensional scheme apolar to $F \in \mathcal{S}_d$.
Then Z contains a scheme evincing a GAD of an extension $F^{\text{ext}} \in \mathcal{S}_D$, for some $D \geqslant d$.

W. Buczyńska, J. Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebraic Geom. 23 (2014), pp. 63–90.

A. Bernardi, J. Brachat, B. Mourrain, *A comparison of different notions of ranks of symmetric tensors*, Linear Algebra Its Appl. 460 (2014), pp. 205–230.

4 Why GADs

Understanding minimal apolar schemes (cactus)

Let Z be a zero-dimensional scheme apolar to $F \in \mathcal{S}_d$.
Then Z contains a scheme evincing a GAD of an extension $F^{\text{ext}} \in \mathcal{S}_D$, for some $D \geqslant d$.

Moreover

If there are integers $0 \leqslant c_i \leqslant d$ such that the i-th connected component of Z is contained in a $(d - c_i + 1)$-fat point, then Z contains a scheme evincing a GAD of F of type

$$F = \sum_{i=1}^{s} L_i^{e_i} Q_i, \quad \text{with } c_i \leqslant e_i \leqslant d \text{ and } Q_i \in \mathcal{S}_{d-e_i}.$$

4 Waring & tangential decompositions

(Minimal) Waring decomposition

Let Z be the scheme evincing a minimal GAD of $F \in \mathcal{S}_d$ of type

$$F = \sum_{i=1}^{s} L_i^d.$$

Then Z is regular in degree d.

A. Bernardi, D. Taufer, *Waring, tangential and cactus decompositions*, J. Math. Pures Appl. 143 (2020), pp. 1–30.

4 Waring & tangential decompositions

(Minimal) Tangential decompositions

Let Z be the scheme evincing a minimal GAD of $F \in \mathcal{S}_d$ of type

$$F = \sum_{i=1}^{s} L_i^{d-1} G_i.$$

Then Z is regular in degree d.

A. Bernardi, D. Taufer, *Waring, tangential and cactus decompositions*, J. Math. Pures Appl. 143 (2020), pp. 1–30.

4 "Balanced" schemes with a small number of independent supports

Let Z be the scheme evincing a GAD of $F \in \mathcal{S}_d$ of type

$$F = \sum_{i=1}^{m} L_i^{e_i} G_i,$$

where $m \leqslant n$ and the L_i's are in general position.

- If $\min_{i \neq j}\{e_i + e_j\} > d$, or
- If $\min_{i \neq j}\{e_i + e_j\} > d - 2$ and Z is irredundant,

then Z is regular in degree d.

Corollary: Minimal Z as above are regular whenever $\forall i \ : \ e_i > \frac{d}{2} - 1$.

4 "Balanced" schemes with a small number of independent supports

Idea

$I(Z)_d^{-1}$	X_1^d	$\dots$	$X_1^{e_1}X_m^{d-e_1}$	$\dots$	$X_m^{e_m}X_1^{d-e_m}$	$\dots$	X_m^d	$\dots$
$X_1^{e_1}G_1$								
$X_1^{e_1+1}\partial G_1$					0			
$\vdots$								
X_1^d								
$\vdots$								
$X_m^{e_m}G_m$								
$X_m^{e_m+1}\partial G_m$			0					
$\vdots$								
X_m^d								

5 Outline

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5 GADs of 0

Let us consider a GAD of type

$$F = \sum_{i=1}^{m} X_i^{e_i} G_i.$$

If $e_i + e_j \leqslant d$, then we can change the GAD:

$$X_i^{e_i} G_i + X_j^{e_j} G_j = X_i^{e_i}(G_i + X_j^{e_j} H) + X_j^{e_j}(G_j - X_i^{e_i} H),$$

for any

$$H \in \mathcal{S}_{d-e_i-e_j}.$$

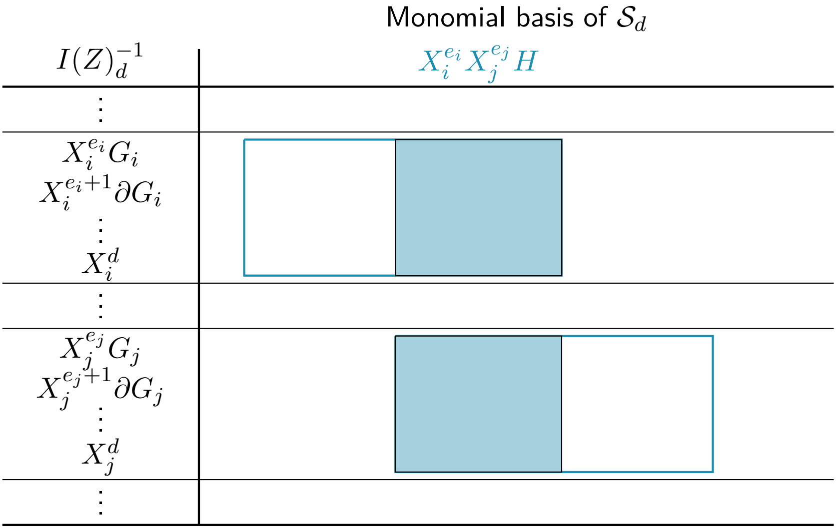
The space GADs of F depends on the GADs of 0 with the same supports/exponents:

$$0 = X_i^{e_i}(X_j^{e_j} H) + X_j^{e_j}(-X_i^{e_i} H).$$

5 GADs of 0

$I(Z)_d^{-1}$	Monomial basis of $\mathcal{S}_d$
$\vdots$	
$X_i^{e_i}G_i$ $X_i^{e_i+1}\partial G_i$ $\vdots$ X_i^d	Coefficients of H
$\vdots$	
$X_j^{e_j}G_j$ $X_j^{e_j+1}\partial G_j$ $\vdots$ X_j^d	Coefficients of H
$\vdots$	

5 GADs of 0



Monomial basis of $\mathcal{S}_d$
$I(Z)_d^{-1}$
$X_i^{e_i} X_j^{e_j} H$
$\vdots$
$X_i^{e_i} G_i$
$X_i^{e_i+1} \partial G_i$
$\vdots$
X_i^d
$\vdots$
$X_j^{e_j} G_j$
$X_j^{e_j+1} \partial G_j$
$\vdots$
X_j^d
$\vdots$

5 Random testing

$$F = x^e G_x + y^e G_y + z^e G_z \in \mathbb{C}[x, y, z]_d.$$

[reg, len]

$d \setminus e$	1	2	3	4
2	$[2, 3 \cdot 2]$	$[1, 3 \cdot 1]$	-	-
3	$[4, 3 \cdot 4]$	$[2, 3 \cdot 2]$	$[1, 3 \cdot 1]$	-
4	$[5, 3 \cdot 6]$	$[4, 3 \cdot 4]$	$[2, 3 \cdot 2]$	$[1, 3 \cdot 1]$
5	$[6, 3 \cdot 9]$	$[5, 3 \cdot 6]$	$[4, 3 \cdot 4]$	$[2, 3 \cdot 2]$
6	$[7, 3 \cdot 12]$	$[6, 3 \cdot 9]$	$[5, 3 \cdot 6]$	$[4, 3 \cdot 4]$
7	$[9, 3 \cdot 16]$	$[7, 3 \cdot 12]$	$[6, 3 \cdot 9]$	$[5, 3 \cdot 6]$
8	$[10, 3 \cdot 20]$	$[9, 3 \cdot 16]$	$[7, 3 \cdot 12]$	$[6, 3 \cdot 9]$

5 Random testing

$$F = x^e G_x + y^e G_y + z^e G_z + u^e G_u \in \mathbb{C}[x, y, z, u]_d.$$

[reg, len]

$d \setminus e$	1	2	3	4
2	$[2, 4 \cdot 2]$	$[1, 4 \cdot 1]$	-	-
3	$[3, 4 \cdot 5]$	$[2, 4 \cdot 2]$	$[1, 4 \cdot 1]$	-
4	$[4, 4 \cdot 8]$	$[3, 4 \cdot 5]$	$[2, 4 \cdot 2]$	$[1, 4 \cdot 1]$
5	$[5, 4 \cdot 14]$	$[4, 4 \cdot 8]$	$[3, 4 \cdot 5]$	$[2, 4 \cdot 2]$
6	$[6, 4 \cdot 20]$	$[5, 4 \cdot 14]$	$[4, 4 \cdot 8]$	$[3, 4 \cdot 5]$
7	$[7, 4 \cdot 30]$	$[6, 4 \cdot 20]$	$[5, 4 \cdot 14]$	$[4, 4 \cdot 8]$
8	$[8, 4 \cdot 40]$	$[7, 4 \cdot 30]$	$[6, 4 \cdot 20]$	$[5, 4 \cdot 14]$

5 Open questions

- Are all the minimal apolar schemes (i.e. cactus) d-regular?
- Is there a minimal apolar schemes (i.e. cactus) that is d-regular?
- Are apolar schemes *generically* d-regular?
- What properties of the given tensor are needed for ensuring d-regularity?
- Finding (large) families of d-regular apolar schemes.
- Constructing apolar schemes of exceptionally high regularity.
- Is d even tight?

Thanks for your attention!

Dziękuję!