The linear span of uniform matrix product states (uMPS)

joint work with Claudia De Lazzari and Tim Seynnaeve

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Overview

1. Tensor Networks
2. Uniform Matrix Product States (uMPS)
3. Results
4. Open Problem

What are Tensor Networks?

vector v_j

matrix M_{ij}

3-index tensor T_{ijk}

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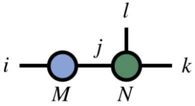

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Figure: Diagrammatic representation of Tensor Contraction

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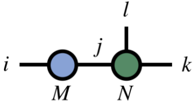

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Matrix Operations via Tensor Networks

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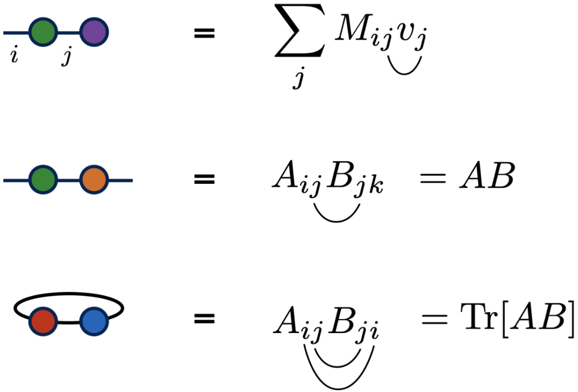

Figure: Matrix operations via Tensor Networks

Motivation from Quantum Physics

Quantum Many-Body system consisting of d particles, each one with the wave function residing in finite dimensional Hilbert space H_i, with $\dim(H_i) = n$ and the orthonormal basis of H_i as $\{|e_{h_i}\rangle\}_{h_i=1}^{n}$.

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The wave function of many-body system is a tensor product of states $H = \bigotimes_{i=1}^{d} H_i$.

$$|\psi\rangle = \sum_{h_1,\dots,h_d=1}^{n} A_{h_1 \cdots h_d} |e_{h_1}\rangle \otimes \cdots \otimes |e_{h_d}\rangle$$

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Figure: One solution: Quantum Computers!

Another Solution: Tensor Networks!

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Figure: Physical Corner of many-body Hilbert Space

Uniform Matrix Product States (uMPS)

Definition

The uniform Matrix Product State parametrization is given by the map

$$\Phi : (\mathbb{C}^{m \times m})^n \to (\mathbb{C}^n)^{\otimes d}$$

$$(A_0, \ldots, A_{n-1}) \mapsto \sum_{0 \leq i_1, \ldots, i_d \leq n-1} \mathrm{Tr}(A_{i_1} \cdots A_{i_d})\, e_{i_1} \otimes \cdots \otimes e_{i_d}.$$

$$\mathrm{uMPS}(m, n, d) = \overline{Im(\Phi)}$$

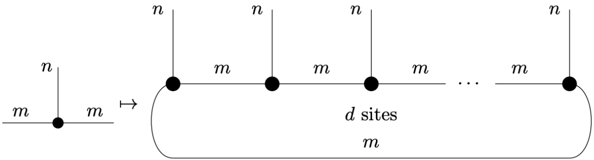

Figure: Graphical representation of uMPS(m, n, d)

Fundamental Questions

What is the dimension of $\langle \mathrm{uMPS}(m,n,d) \rangle$?

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What is the dimension of $h\mathrm{uMPS}(m; n; d)i$?

For which parameters $m; n; d$ 2 N does $h\mathrm{uMPS}(m; n; d)i$ fill the ambient space?

Observations

$$\mathrm{Cyc}^d(\mathrm{C}^n) := f\,!\;\; 2\;(\mathrm{C}^n)\;\;{}^d\;j\qquad !\;=\;!\quad 8\quad 2\;C_d g{:}$$

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$$\mathrm{Cyc}^d(\mathbb{C}^n) := \{\omega \in (\mathbb{C}^n)^{\otimes d} \mid \sigma\omega = \omega \quad \forall \sigma \in C_d\}.$$

Observation 1

The set $\mathrm{uMPS}(m,n,d) \subseteq \mathrm{Cyc}^d(\mathbb{C}^n)$. Given $M_1, \ldots, M_d \in \mathbb{C}^{m \times m}$, $\sigma \in C_d$

$$\mathrm{Tr}(M_1 \cdots M_d) = \mathrm{Tr}(M_{\sigma(1)} \cdots M_{\sigma(d)}).$$

Observations

$$\mathrm{Dih}^d(\mathbb{C}^n) := \{ \omega \in (\mathbb{C}^n)^{d} \mid \omega = \omega \quad \forall \quad \in D_{2d} \}.$$

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Observation 2 ([Gre14, Theorem 1.1])

$\mathrm{uMPS}(2;2;d)\qquad \mathrm{Dih}^d(\mathrm{C}^n)$. Given $A_{i_1};\qquad ;A_{i_d}\;2\;\mathrm{C}^{2\quad 2}$

$$\mathrm{Tr}(A_{i_1}\qquad A_{i_d}) = \mathrm{Tr}(A_{i_d}\qquad A_{i_1}):$$

Idea 1 - Cayley-Hamilton Technique

Cayley-Hamilton Technique

Let $c = (c_1, \ldots, c_s) \in \mathbb{C}^s$ be a vector of coefficients and $\{i^j_\ell\}_{1 \le \ell \le d, 1 \le j \le s}$ be indices, with $i^j_\ell \in [n]$. Assume that for every n-tuple $(A_0, \ldots, A_{n-1})$ of $m \times m$ matrices and every $k < m$ the following identity holds:

$$\sum_{j=1}^{s} c_j \,\mathrm{Tr}(A_{i^j_1} \cdots A_{i^j_d} A_0^k) = 0.$$

Then the same identity holds for arbitrary $k \in \mathbb{N}$.

Example - 2 2 matrices

The following identity holds for any 2 2 matrices $A_0; A_1; A_2; A_3$ and k 0:

$$\mathrm{Tr}(A_1A_2A_0A_3A_0^k) + \mathrm{Tr}(A_2A_3A_0A_1A_0^k) + \mathrm{Tr}(A_3A_1A_0A_2A_0^k)$$
$$= \mathrm{Tr}(A_1A_0A_2A_3A_0^k) + \mathrm{Tr}(A_2A_0A_3A_1A_0^k) + \mathrm{Tr}(A_3A_0A_1A_2A_0^k):$$

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$$\mathrm{Tr}(A_1A_2A_0A_3) + \mathrm{Tr}(A_2A_3A_0A_1) + \mathrm{Tr}(A_3A_1A_0A_2)$$
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Strict Containment

Theorem 1

If n 3 and d $\frac{(m+1)(m+2)}{2}$, then uMPS$(m; n; d)$ is contained in a proper linear subspace of the space of cyclically invariant tensors.

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Sketch of Proof:

> We need to find a non-trivial linear relation between traces of matrices which is not given by cyclic permutation.

Strict Containment

Theorem 1

If $n \geq 3$ and $d \geq \frac{(m+1)(m+2)}{2}$, then $\mathrm{uMPS}(m; n; d)$ is contained in a proper linear subspace of the space of cyclically invariant tensors.

Sketch of Proof:

- We need to find a non-trivial linear relation between traces of matrices which is not given by cyclic permutation.
- We proved the following technical identity using Cayley-Hamilton trick

$$\sum_{\sigma \in S_m,\ \tau \in C_{m+1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\,\mathrm{Tr}(A_{\sigma(0)} B^{\tau(0)} \cdots A_{\sigma(m-1)} B^{\tau(m-1)} A_{\sigma(m)} B^{\ell}) = 0.$$

Idea 2 - Trace Parametrization

A *bracelet* (of length d on the alphabet $[n]$) is an equivalence class of words, where two words are equivalent if they agree up to the action D_{2d}.

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Technical Lemma

For every bracelet $b = (b_1, \ldots, b_k)$, there is a unique polynomial

$$P_b(T_0, T_1, T_{00}, T_{01}, T_{11}) \in \mathbb{C}[T_0, T_1, T_{00}, T_{01}, T_{11}]$$

such that for every pair (A_0, A_1) of 2×2 matrices, the following equality holds:

$$\mathrm{Tr}(A_{b_1} \cdots A_{b_k}) = P_b(\mathrm{Tr}(A_0), \mathrm{Tr}(A_1), \mathrm{Tr}(A_0^2), \mathrm{Tr}(A_0 A_1), \mathrm{Tr}(A_1^2)).$$

Idea 2 - Trace Parametrization

$\mathrm{uMPS}(2, 2, d)$ is the image of the polynomial map

$$: \mathbb{C}^5 \to \mathrm{Dih}^d(\mathbb{C}^2)$$
$$(T_0, T_1, T_{00}, T_{01}, T_{11}) \mapsto \sum_b P_b(T_0, T_1, T_{00}, T_{01}, T_{11}) e_b,$$

where b runs over all bracelets of length d.

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where b runs over all bracelets of length d.

This is the trace parametrization of uMPS$(2, 2, d)$.

Bound on dimensions

Theorem 2

For every $d \in \mathbb{N}$, we have the inequality

$$\dim\langle \mathrm{uMPS}(2,2,d)\rangle \quad \begin{cases} \frac{1}{192}(d+6)(d+4)^2(d+2) & \text{for } d \text{ even;} \\ \frac{1}{192}(d+7)(d+5)(d+3)(d+1) & \text{for } d \text{ odd.} \end{cases}$$

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Sketch of Proof:

Using the trace parametrization of $\mathrm{uMPS}(2,2,d)$, $\dim\langle \mathrm{uMPS}(2,2,d)\rangle$ is at most the number of degree d monomials in $\mathbb{C}[T_0, T_1, T_{00}, T_{01}, T_{11}]$.

Open Problem

Conjecture

$$\dim h uMPS(2;2;d)i = \begin{cases} \frac{1}{192}(d^4 \quad 4d^2 + 192d + 192) & \text{for } d \text{ even,} \\ \frac{1}{192}(d^4 \quad 10d^2 + 192d + 201) & \text{for } d \text{ odd.} \end{cases}$$

References

The work presented here is based on:

- De Lazzari C, Motwani HJ, Seynnaeve T. **The linear span of uniform matrix product states.** arXiv preprint arXiv:2204.10363. 2022 Apr 21.

- Code available at: `https://github.com/harshitmotwani2015/uMPS/`

Pictures of Tensor Networks are taken from `https://tensornetwork.org/`

Additional Reference

- Poulin D, Qarry A, Somma R, Verstraete F. Quantum simulation of time-dependent Hamiltonians and the convenient illusion of Hilbert space. Physical review letters. 2011 Apr 29;106(17):170501.
- Greene J. Traces of matrix products. The Electronic Journal of Linear Algebra. 2014 Jan 1;27:716-34.