Toric degenerations of flag varieties and combinatorial mutations

Francesca Zaffalon

23 September 2022

Ghent University and KU Leuven
Kickoff workshop - AGATES

joint work with Oliver Clarke and Fatemeh Mohammadi

Motivation

- General algebraic varieties can be difficult to study.
- Toric varieties are algebraic varieties modeled on convex polytopes.
- Significant algebraic properties of toric varieties can be read off from the associated polytope:

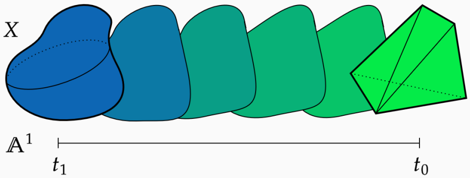
dimension	$\longleftrightarrow$	dimension of the polytope
degree	$\longleftrightarrow$	volume
Hilbert polynomial	$\longleftrightarrow$	Ehrhart poynomial

How can we use toric varieties to study general algebraic varieties?

Toric degenerations

A **toric degeneration** of a variety X is a flat family $\mathcal{F} \to \mathbb{A}^1$ such that:

- the fiber $\mathcal{F}_t$ over $t \in \mathbb{A}^1 \setminus \{0\}$ is isomorphic to X;
- the fiber $\mathcal{F}_0$ over 0 is a toric variety.



- Toric degenerations have been studied in algebraic geometry, representation theory, cluster algebra, and tropical geometry.
- The geometric invariants of X can be read from any fiber in the degeneration, in particular from the toric fiber.

Two main questions

When studying toric degeneration there are two main questions:

Question 1. How do we construct toric degenerations of a given variety?

Question 2. What are the relations between two different toric degenerations of the same variety? Which algebraic properties are preserved under degenerations?

Grassmannian and flag varieties

We want to answer to **Question 1** and **Question 2** for Grassmannian and flag varieties.

- The **Grassmannian** $\mathrm{Gr}(k, n)$ is the variety of k-dimensional linear subspaces in $\mathbb{K}^n$.
- The **flag variety** $\mathcal{F}\ell_n$ is the variety of flags $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$, where $V_k \in \mathrm{Gr}(k, n)$. The flag variety naturally lives in a product of Grassmannians:

$$\mathcal{F}\ell_n \subseteq \mathrm{Gr}(1, n) \times \cdots \times \mathrm{Gr}(n-1, n)$$

Plücker variables

- $\mathrm{Gr}(k,n)$ can be embedded in a projective space via the **Plücker coordinates**:

$$\mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

where coordinates of $\mathbb{P}^{\binom{n}{k}-1}$ are labeled by k-subsets of $[n]$.

$$p_I = \det X[I] \text{ for } I \in \binom{[n]}{k}$$

- $\mathcal{F}\ell_n$ can be embedded into a product of projective spaces $\mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$, where coordinates are labeled by subsets of $[n]$:

$$p_I = \det X[I] \text{ for } I \subseteq [n]$$

Question 1. How do we construct toric degenerations of a given variety?

Gröbner degenerations

- A classical way is via **Gröbner degenerations**.
- Let $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ be an ideal. Given $w \in \mathbb{R}^{n+1}$ we can define the ideal

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle$$

  where

$$\mathrm{in}_w(f) = \sum_{\alpha \cdot w \text{ minimal}} f_\alpha x^\alpha.$$

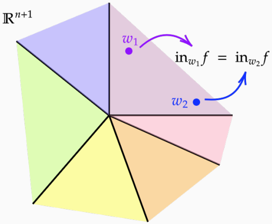
  Example. Let
  $f = p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \in \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$. Then

  - for $w = (1, 0, 0, 0, 0, 1)$ we have $\mathrm{in}_w(f) = -p_{13}p_{24} + p_{14}p_{23}$;
  - for $w = (1, 1, 1, 2, 3, 4)$ we have $\mathrm{in}_w(f) = p_{14}p_{23}$.

- It is possible to generate a flat family of varieties over $\mathbb{A}^1$ such that the special fiber corresponds to the ideal $\mathrm{in}_w(I)$.
- If $\mathrm{in}_w(I)$ is a toric ideal, we have a toric degeneration.

Gröbner fan

- The **Gröbner fan** of $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is a fan in $\mathbb{R}^{n+1}$ where w_1 and w_2 lie in the same cone if and only if they give the same initial ideal.



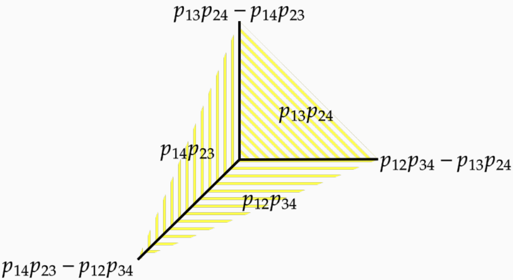
- Not every point in the Gröbner fan gives a toric degeneration: a generic weight $w \in \mathbb{R}^{n+1}$ give rise to a monomial ideal $\text{in}_w(I)$.
- $\text{in}_w(I)$ needs to be binomial and prime

$$\Downarrow$$

We restrict to the w in the fan such that $\text{in}_w(I)$ contains no monomial.

Gröbner fan of $\mathrm{Gr}(2,4)$

Example. Consider $\mathrm{Gr}(2,4) = V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$. The Gröbner fan consists of 7 cones:



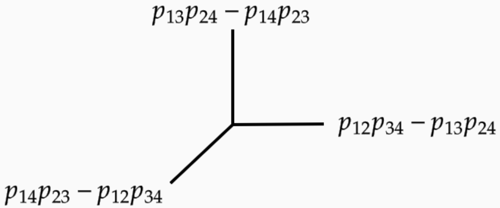
Idea: restrict to $\{w \in \mathbb{R}^{n+1} \mid \min\{\alpha \cdot w \mid f_\alpha \neq 0\}$ is achieved at least twice$\}$.

Tropicalization

This space is the **tropicalization** of $X = V(I)$:

$$\text{trop}(X) = \bigcap_{f \in I} \{w \in \mathbb{R}^{n+1} \mid \min\{\alpha \cdot w \mid f_\alpha \neq 0\} \text{ is achieved at least twice}\}$$

Example. For Gr(2, 4) we get 3 top-dimensional cones. All of them give rise to toric degenerations of Gr(2, 4).

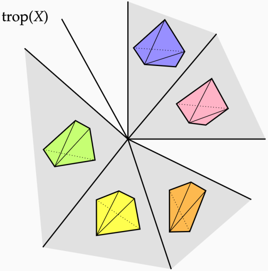


Tropicalization and toric degenerations

Moreover $\mathrm{in}_w(I)$ needs to be binomial and prime.

$$\Downarrow$$

We restrict to the cones giving prime initial ideals, which we call prime cones.



Tropicalization of Grassmannian and flag varieties

Computing points in top-dimensional cones of the tropicalization of a variety is not trivial:

- trop(Gr(3, 6)) is a 3-dimensional fan with 1005 maximal cones. They merge into 7 symmetry classes, 6 of which give non-isomorphic toric degenerations.
- trop(Gr(3, 7)) is a 5-dimensional fan with 252000 maximal cones. They merge into 125 cones modulo S_7, 69 of which give non-isomorphic toric degenerations.
- trop($\mathcal{F}\ell_5$) has 69780 maximal cones, 536 modulo the action of $S_5 \rtimes \mathbb{Z}_2$. 180 give toric degenerations.

Points in the tropicalization of the Grassmannian

We want ways to generate points in the tropicalization of these varieties.

For Grassmannian and flag varieties we can do this using **matching fields**:

Let $\sigma \in S_n$ and consider the matrix

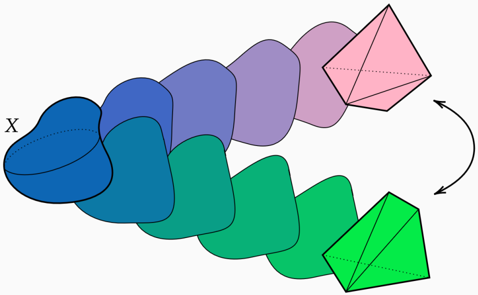
$$M^\sigma = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ Nn & N(n-1) & \dots & N \\ \vdots & \vdots & & \vdots \\ N^{k-2}(n-1) & N^{k-2}(n-1) & \dots & N^{k-2} \end{pmatrix}$$

We will show that the vector w obtained by computing the Plücker coordinates of M^σ give rise to a toric degeneration of $\mathrm{Gr}(k,n)$, in particular $w \in \mathrm{trop}(\mathrm{Gr}(k,n))$.

Question 2. What are the relations between two different toric degenerations of the same variety?

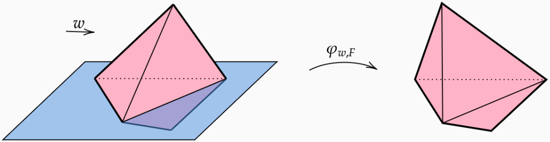
Relations between toric degenerations

- Isomorphic toric degenerations correspond to lattice equivalent polytopes.
- What is the relation between polytopes corresponding to non-isomorphic toric degenerations?



Combinatorial mutations: Definition

- **Combinatorial mutations** are piecewise-linear functions on polytopes:



This is defined as a tropical map

$$\varphi_{w,F} : x \mapsto x - \min\{\langle x, v \rangle \mid v \in F\} w$$

- Combinatorial mutation equivalent polytopes are not lattice equivalent. They have same volume and Ehrhart polynomial.

Theorem. The polytopes of the matching fields M^{σ} are combinatorial mutation equivalent for $\sigma \in S_n$ and they all give rise to toric degenerations of $\mathrm{Gr}(k, n)$

Idea of proof.

- The matching field corresponding to $\sigma_{GT} = (n \ n-1 \ \dots \ 1)$ gives a toric degeneration.
- All the polytopes associated to matching fields are combinatorial mutation equivalent to the polytope of M^{σ}_{GT}.
- A matching field inherits the property of giving rise to a toric degeneration from another matching field whenever the two polytopes are combinatorial mutation equivalent.

Computational results

Using matching fields and combinatorial mutations we get toric degenerations of the Grassmannian and flag varieties. In particular we have:

- for $\mathrm{Gr}(3,6)$ we obtain all the 6 possible toric degenerations.
- for $\mathrm{Gr}(3,7)$ we obtain 40 out of 69 of the possible toric degenerations.
- for $\mathcal{F}\ell_4$ we obtain all the 4 possible toric degenerations.

Oliver Clarke, Fatemeh Mohammadi, Francesca Zaffalon
Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes

Open problems

- How can we construct larger families of toric degenerations of Grassmannian and flag varieties?
- How can we construct toric degenerations of other varieties (for example secant varieties)?

Thank you!