# Persistent Tensors:
# Multipartite Entanglement & Algebraic Geometry

*Masoud Gharahi*

AGATES: Tensors from the Physics viewpoint
04-07 October 2022, Warsaw, Poland

# Outline

## I. Entanglement Classification

- $\ell$-multirank
- Secant Varieties
- Tensor Rank & Border Rank

**Example:** **4-Qubit Entanglement**

## II. Entanglement Transformation

- SLOCC Transformation (Restriction)
- Asymptotic SLOCC transformation (Degeneration)
- Persistent Tensors

**Example:** $|\,\mathrm{GHZ}(d,n)\rangle \xrightarrow{\text{SLOCC}} |\,\mathrm{M}(d,n)\rangle$

# Motivation

Motivation: Two central problems:

i) To know which states are equivalent in the sense that they are capable of performing the same tasks in quantum information processing.
ii) Interconversion between different resources by SLOCC and asymptotic SLOCC.

- SLOCC: equivalency based on *local invertible transforamations*

$$|\Psi\rangle \sim |\Phi\rangle \qquad (0 < \text{Prob.} \leq 1) \qquad \text{iff} \qquad |\Psi\rangle = \mathrm{SL}_1 \otimes \cdots \otimes \mathrm{SL}_n |\Phi\rangle$$

- Using Schmidt rank, one can characterize SLOCC convertibility of bipartite systems

$$|\psi\rangle \xrightarrow{\text{SLOCC}} |\varphi\rangle \quad \Leftrightarrow \quad \mathrm{rk}_S(\psi) \geq \mathrm{rk}_S(\varphi)$$

Qustion: How can we

1. classify multipartite entanglement?
2. characterize SLOCC convertibility of multipartite systems?

# Entanglement Classification

# $\ell$-Multiranks I

Flattening: By defining an ordered $\ell$-tuples $I = (i_1, i_2, \ldots, i_\ell)$, and an ordered $(n-\ell)$-tuples $\bar{I}$ such that $I \cup \bar{I} = (1, 2, \ldots, n)$, we can reshape the $n$-fold tensor product space (here, $\mathcal{H}_n = \otimes_{i=1}^{n} \mathbb{C}^2$) to a matrix

$$\mathcal{H}_n \simeq \mathcal{H}_I \otimes \mathcal{H}_{\bar{I}}\,, \qquad \mathcal{H}_I = \mathbb{C}^{2^\ell}\,, \quad \mathcal{H}_{\bar{I}} = \mathbb{C}^{2^{n-\ell}}\,.$$

Using Dirac notation, the flattening of $|\psi\rangle$ reads

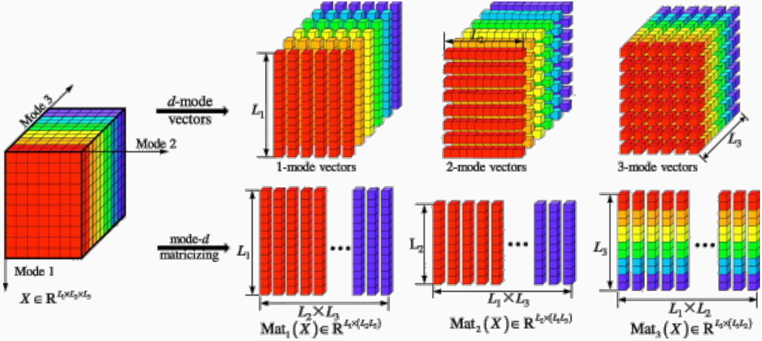
$$\mathcal{M}_I[\psi] = \left(\langle e_0|\psi\rangle, \langle e_1|\psi\rangle, \ldots, \langle e_{2^\ell-1}|\psi\rangle\right)^{\mathrm{T}}\,, \quad \text{Matrix Order} = 2^\ell \times 2^{n-\ell}\,,$$

where $|e_j\rangle \equiv |j\rangle$, with $j \in \{0,1\}^\ell$, canonical basis of $\mathcal{H}_I$.

$\ell$-multirank is a $\binom{n}{\ell}$-tuple of ranks of matrices $\mathcal{M}_I[\psi]$.

# $\ell$-Multiranks II

Example: 3-fold tensor product space




Flattening of a 3-fold tensor product space to three different matrices [https://doi.org/10.1016/j.isprsjprs.2013.06.001].

The one-multirank is then a triple $(r_1, r_2, r_3)$.

# $\ell$-Multiranks III

Proposition. $\ell$-multiranks are SLOCC invariants.

*Proof.* SLOCC equivalent states, i.e. $|\tilde{\psi}\rangle = (\otimes_{i=1}^{n} A_i)\,|\psi\rangle$, where $|\psi\rangle \in \mathcal{H}$ and $A_i \in \mathrm{SL}(d, \mathbb{C})$, yield

$$\mathcal{M}_I[\tilde{\psi}] = (\otimes_{i \in I} A_i)\, \mathcal{M}_I[\psi]\, \left(\otimes_{i \in \bar{I}} A_i\right)^{\mathrm{T}}$$

∎

A state is genuinely entangled iff all $\ell$-multiranks are greater than one.

A given $\ell$-multirank configuration determines a determinantal variety in the projective Hilbert space $\mathbb{P}\mathcal{H}$.

The determinantal variety is a subset of all matrices with rank $r$ or less in $\mathbb{P}\mathcal{H}$, that is just the common zero locus of the $(r+1) \times (r+1)$ minors.

# Secant Varieties I

Consider $n$-qubit states: $$|\psi\rangle = \sum_{i\in\{0,1\}^n} \mathfrak{c}_i |i\rangle$$

The space of fully separable states $|\psi\rangle$ has the structure of a Segre variety which is embedded in the ambient space as follows

$$\Sigma_{\mathbf{1}}^n : \ \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2^n-1}$$

- The $k$-secant of a variety joins its $k$ points, so it can liaise to the concept of *quantum superposition*.

$k$-secant variety $\sigma_k(\Sigma_{\mathbf{1}}^n :) \equiv$ union of $k$-secants of the Segre variety

$\Rightarrow$ it corresponds to the set of all entangled states arising from the superposition of $k$ fully separable states.

# Secant Varieties II

**Proposition.** $k$-secant varieties are SLOCC invariants.

*Proof.* If the points of variety $\mathcal{X}$ remains invariant under the action of a group $G$ then so is any its auxiliary variety which is built from points of $\mathcal{X}$. ■

- The higher $k$-secant variety fills the ambient space $\mathbb{P}(\mathbb{C}^{2^{\otimes n}})$ when

$$k = \left\lceil \frac{2^n}{n+1} \right\rceil$$

This $k$ indicates the number of entanglement families.

The *proper $k$-secant* is the set $\sigma_k(\Sigma_{\mathbf{1}}^n) \setminus \sigma_{k-1}(\Sigma_{\mathbf{1}}^n)$.

# Tensor Rank & Border Rank

Tensor rank: The rank of a tensor $\mathcal{T} \in V_1 \otimes \cdots \otimes V_n$ is defined as the minimum number of simple tensors (fully separable states) that sum to $\mathcal{T}$.

$$\mathrm{rk}(\mathcal{T}) = \min \left\{ r \middle| \mathcal{T} = \sum_{p=1}^{r} v_1^{(p)} \otimes \cdots \otimes v_n^{(p)}\,,\ \text{for some } v_i^{(p)} \in V_i \right\}.$$

Border rank: The border rank of $\mathcal{T}$ is the smallest $r$ such that $\mathcal{T}$ is a limit of tensors of rank $r$, i.e.,

$$\mathrm{brk}(\mathcal{T}) = \min \left\{ r \middle| \mathcal{T} = \lim_{\varepsilon \to 0} \mathcal{T}_\varepsilon\,, \text{s.t. } \forall \varepsilon\ \ \mathrm{rk}(\mathcal{T}_\varepsilon) = r \right\}.$$

# 3 Qubits & 3 Qutrits

2-Secant:

$$|\,\mathrm{GHZ}(2,3)\rangle = |000\rangle + |111\rangle \qquad \mathrm{rk}(\mathcal{G}(2,3)) = 2$$

While $|\,\mathrm{W}_3\rangle = |001\rangle + |010\rangle + |100\rangle$ has tensor rank 3, its border rank is 2 since

$$|\,\mathrm{W}_3\rangle = \lim_{\varepsilon\to 0} \frac{\left((|0\rangle + \varepsilon|1\rangle)^{\otimes 3} - |000\rangle\right)}{\varepsilon}$$

3-secant:

$$|\,\mathrm{GHZ}(3,3)\rangle = |000\rangle + |111\rangle + |222\rangle \qquad \mathrm{rk}(\mathcal{G}(3,3)) = 3$$

While $|\,\mathrm{Y}_3\rangle = |002\rangle + |020\rangle + |200\rangle + 2(|011\rangle + |101\rangle + |110\rangle)$ has tensor rank 5, its border rank is 3 since

$$|\,\mathrm{Y}_3\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left((|0\rangle + \frac{\varepsilon}{\sqrt{2}}|1\rangle + \varepsilon^2|2\rangle)^{\otimes 3} + (|0\rangle - \frac{\varepsilon}{\sqrt{2}}|1\rangle)^{\otimes 3} - 2|000\rangle\right)$$

# Classification Algorithm

We use $k$-secant varieties and $\ell$-multiranks as the SLOCC invariants to group orbits (classes) into finite number of families and subfamilies. Hence, the classification algorithm can be summarized as:

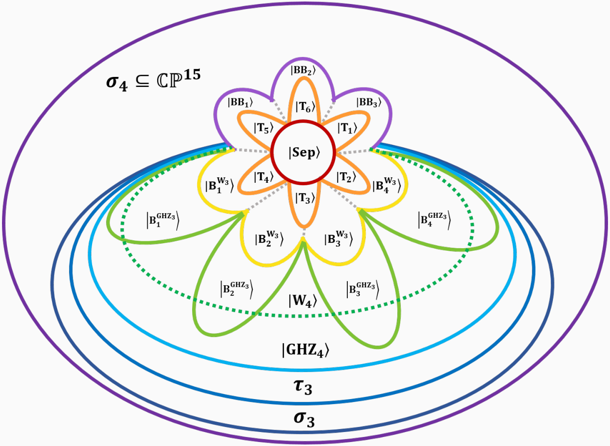
1. find families by identifying $\Sigma_{\mathbf{1}}^n$, $\sigma_2(\Sigma_{\mathbf{1}}^n), \ldots, \sigma_k(\Sigma_{\mathbf{1}}^n)$
2. split families to secants and tangents by identifying $\tau_2(\Sigma_{\mathbf{1}}^n), \ldots, \tau_k(\Sigma_{\mathbf{1}}^n)$;
3. find subfamilies by identifying $\ell$-multiranks

# 4-Qubit Entanglement (I)

Fine-structure classification of four-qubit entanglement

| $\boldsymbol{\Sigma_1^4}$ | $\sigma_2$ | $\tau_2$ | $\sigma_3$ | $\tau_3$ | $\sigma_4$ |
|---|---|---|---|---|---|
| $\|\mathrm{Sep}\rangle$ | $\|\mathrm{GHZ}_4\rangle$ | $\|\mathrm{W}_4\rangle$ | $\|(333)\rangle$ | $\|(333)'\rangle$ | $\|(444)\rangle$ |
| | $\|\mathrm{B}_i^{\mathrm{GHZ}_3}\rangle_{i=1}^4$ | $\|\mathrm{B}_i^{\mathrm{W}_3}\rangle_{i=1}^4$ | $\|(332)\rangle$ | $\|(332)'\rangle$ | $\|(443)\rangle$ |
| | $\|\mathrm{T_i}\rangle_{i=1}^6$ | | $\|(323)\rangle$ | $\|(323)'\rangle$ | $\|(434)\rangle$ |
| | | | $\|(233)\rangle$ | $\|(233)'\rangle$ | $\|(344)\rangle$ |
| | | | | | $\|(442)\rangle$ |
| | | | | | $\|(424)\rangle$ |
| | | | | | $\|(244)\rangle$ |
| | | | | | $\|\mathrm{BB}_i\rangle_{i=1}^3$ |

# 4-Qubit Entanglement (II)




Petal-like classification of four-qubit entanglement using secant and tangent varieties. From the outer classes, one can go to the inner ones by noninvertible SLOCC (from $\sigma_k$ to $\tau_k$ also in an approximate way), thus generating the entanglement hierarchy. Based on this figure, one can also figure out the classification of $n$-qubit entanglement with the $(n-1)$-qubit entanglement classification as its core part. This reminds the Mendeleev classification.

# Entanglement Transformation

# SLOCC Transformation

SLOCC transformation: Let $|\psi\rangle \in U_1 \otimes \cdots \otimes U_n$ and $|\varphi\rangle \in V_1 \otimes \cdots \otimes V_n$. We say that $|\psi\rangle$ can be transformed into $|\varphi\rangle$ via SLOCC (denoted by $|\psi\rangle \xrightarrow{\text{SLOCC}} |\varphi\rangle$) if there exist linear transformations $A_i : U_i \to V_i$ such that

$$(\otimes_{i=1}^{n} A_i)|\psi\rangle = |\varphi\rangle \,.$$

• It is well known that a GHZ state cannot be transformed into a W state by SLOCC, as they belong to distinct entanglement classes, but one can asymptotically produce a W-equivalent state from a GHZ-equivalent state with an arbitrary precision.

# Degeneration

Degeneration: Let $|\psi\rangle \in U_1 \otimes \cdots \otimes U_n$ and $|\varphi\rangle \in V_1 \otimes \cdots \otimes V_n$. We say that $|\psi\rangle$ degenerates into $|\varphi\rangle$ via SLOCC (denoted by $|\psi\rangle \overset{\text{SLOCC}}{\dashrightarrow} |\varphi\rangle$) if there exist linear transformations $A_i(\varepsilon) : U_i \to V_i$ depending polynomially on $\varepsilon$ such that

$$(\otimes_{i=1}^n A_i(\varepsilon))|\psi\rangle = \varepsilon^d |\varphi\rangle + O(\varepsilon^{d+1}),$$

for some $d \in \mathbb{N}$.

Indeed, if $|\psi\rangle \overset{\text{SLOCC}}{\dashrightarrow} |\varphi\rangle$, then $|\varphi\rangle$ can be approximated to arbitrary precision by restrictions of $|\psi\rangle$, i.e.,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^d} (\otimes_{i=1}^n A_i(\varepsilon))|\psi\rangle = |\varphi\rangle .$$

Rate of asymptotic SLOCC transformation from $|\psi\rangle$ into $|\varphi\rangle$:

$$\omega(|\psi\rangle, |\varphi\rangle) = \lim_{\ell \to \infty} \frac{1}{\ell} \inf \left\{ k \in \mathbb{N} \middle| |\psi\rangle^{\boxtimes k} \xrightarrow{\text{SLOCC}} |\varphi\rangle^{\boxtimes \ell} \right\} .$$

# Concise Tensors

Concise tensor: A tensor $\mathcal{T} \in V_1 \otimes \cdots \otimes V_n$ is called 1-concise, i.e., concise in $V_1$, if $\mathcal{T} \notin V_1' \otimes V_2 \otimes \cdots \otimes V_n$ with $V_1' \subsetneq V_1$. Conciseness in other factors is defined analogously. A tensor is called concise if it is $i$-concise for all $i \in \{1, \ldots, n\}$.

- It means that the tensor $\mathcal{T}$ uses all dimensions of the local spaces.

Lemma: Let $\mathcal{T} \in V_1 \otimes \cdots \otimes V_n$ be an $i$-concise tensor. If

$$\mathcal{T} = \sum_{p=1}^{r} v_1^{(p)} \otimes \cdots \otimes v_n^{(p)} ,$$

is a tensor rank decomposition of $\mathcal{T}$, then $\mathrm{Span}\{v_i^{(1)}, \ldots, v_i^{(r)}\} = V_i$.

# Substitution Method

Lemma: Let $\mathcal{T} \in V_1 \otimes \cdots \otimes V_n$ be an $i$-concise tensor. For every subspace $V_i' \subsetneq V_i$ there exists a projection $\pi_i \colon V_i \to V_i'$ such that

$$\mathrm{rk}(\mathcal{T}) - \mathrm{rk}(\pi_i \mathcal{T}) \geq \dim V_i - \dim V_i' ,$$

where $\pi_i \mathcal{T}$ denotes the application of $\pi_i$ on the $i^{\text{th}}$ factor of the tensor $\mathcal{T}$.

*Proof.* Suppose $\mathcal{T} = \sum_{p=1}^{r} v_1^{(p)} \otimes \cdots \otimes v_n^{(p)} \Rightarrow \mathrm{Span}\{v_i^{(1)}, \ldots, v_i^{(r)}\} = V_i$. Thus there exists a subset $S_i \subset \{v_i^{(1)}, \ldots, v_i^{(r)}\}$ consisting of $c_i = \dim V_i - \dim V_i'$ vectors such that $W_i = \mathrm{Span}\{S_i\}$ is complementary to $V_i'$. Consider the projection $\pi_i$ onto $V_i'$ along $W_i$. Applying it to the $i$-th factor of each summand of the rank decomposition we obtain a decomposition of $\pi_i \mathcal{T}$ with at most $r - c_i$ summands, because the summands containing vectors from $S_i$ are sent to 0.

# Persistent Tensors I

Persistent tensor: We define **persistent tensors** inductively.

(i) A tensor $\mathcal{P} \in V_1 \otimes V_2$ is persistent if it is 1-concise.

(ii) A tensor $\mathcal{P} \in V_1 \otimes \cdots \otimes V_n$ with $n > 2$ is persistent if it is 1-concise and there exists a vector $|e\rangle \in V_1$ such that for every covector $\langle f| \in V_1^\vee$ so as $\langle f|e\rangle \neq 0$ the contraction $\langle f|\mathcal{T} \in V_2 \otimes \cdots \otimes V_n$ is a persistent tensor.

Lemma: If $\mathcal{P} \in V_1 \otimes \cdots \otimes V_n$ is a persistent tensor, then for every basis $\{|e_j\rangle | j \in \mathbb{Z}_{d_1}\}$ of $V_1$ the representation

$$\mathcal{P} = \sum_{j=0}^{d_1 - 1} |e_j\rangle \otimes \mathcal{P}_j$$

satisfies the following conditions:

(i) All $\mathcal{P}_j$ are nonzero.

(ii) If $n > 2$, at least one of $\mathcal{P}_j$ is persistent.

# Persistent Tensors II

Examples (non-persistent tensors):

1. The diagonal tensor $\mathcal{G}(d,n)$ (correspondingly, $n$-qudit GHZ state) is not a persistent tensor for $n > 2$ and $d \geq 2$. We have

$$\mathcal{G}(d,n) = \sum_{j=0}^{d-1} |j\rangle \otimes \mathcal{T}_j, \quad \mathcal{T}_j = |j\rangle^{\otimes(n-1)}$$

   where all $\mathcal{T}_j$ are not concise and therefore not persistent.

2. Dicke state$|\mathrm{D}_4^2\rangle = |0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle$ is not a persistent tensor because if we take basis of $V_1$ as $\{|\mathrm{f}_\pm\rangle = |0\rangle \pm |1\rangle\}$, then $\langle \mathrm{f}_\pm | \mathrm{D}_4^2 \rangle = |\,\mathsf{W}_3\rangle \pm |\overline{\mathsf{W}}_3\rangle \equiv |\,\mathsf{GHZ}(2,3)\rangle$ which is not a persistent tensor.

# Persistent Tensors III

Examples (persistent tensors):

1. $\mathcal{W}_n$ is a persistent tensor because for every basis in any local spaces, there is a slice which is equivalent to $\mathcal{W}_{n-1}$ and by induction hypothesis one will arrive to $\mathcal{W}_2 = |01\rangle + |10\rangle$ which is a persistent tensor.
2. $(\frac{\alpha}{2})^2|0011\rangle + (\frac{\beta}{2})^2|0101\rangle + (\frac{\alpha-\beta}{2})^2|0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle$. Actually, for every basis in any local spaces, there is a slice which is equivalent to $|\mathrm{W}_3\rangle$ because the tangle for that slice is zero.
3. $n$-qutrit Y state, i.e., $|\mathrm{Y}_n\rangle = \sum_{\mathfrak{p}\in\mathfrak{S}_n} \mathfrak{p}\{|0\rangle^{\otimes(n-2)}(|02\rangle + |11\rangle)\}$, is a persistent tensor.
4. As a generalization, the following $n$-qudit $\mathrm{Z}$ state is a persistent tensor

$$|\mathrm{Z}(d,n)\rangle = \sum_{j_1+\cdots+j_n=d-1} |j_1\cdots j_n\rangle\,,$$

where $\{|j_1\cdots j_n\rangle | j_i \in \mathbb{Z}_d\}$ denotes the standard basis of $\otimes^n\mathbb{C}^d$.

# Multiqudit M States I

Theorem: If $\mathcal{P} \in V_1 \otimes \cdots \otimes V_n$ is a persistent tensor and $d_k = \dim V_k$, then

$$\mathrm{rk}(\mathcal{P}) \geq \sum_{k=1}^{n-1} d_k - n + 2\,.$$

Corollary: The tensor rank of $n$-qubit W state and $n$-qutrit Y state are $n$ and $2n-1$, respectively.

Corollary: The tensor rank of the following $n$-qudit M state in $\otimes^n \mathbb{C}^d$,

$$|\,\mathrm{M}(d,n)\rangle = \sum_{\mathfrak{p} \in \mathfrak{S}_n} \mathfrak{p} \Big\{ \sum_{\ell=0}^{\lfloor \frac{d-1}{2} \rfloor} |0\rangle^{\otimes(n-2)} |\ell\rangle |d-\ell-1\rangle \Big\}\,,$$

is $\mathrm{rk}(\mathcal{M}(d,n)) = (n-1)d - n + 2$.

Multiqudit M state is a generalization of multiqubit W state within multiqudit systems. They are in the orbit closure of multiqudit GHZ states.

# Multiqudit M States II

Lemma: The border rank of $n$-qudit M state is $\mathrm{brk}(\mathcal{M}(d,n)) = d$.

*Proof.* Regarding Z states, we can rewrite the $n$-qudit M state as follows

$$|\,\mathrm{M}(d,n)\rangle = \sum_{\mathfrak{p}\in\mathfrak{S}_n} \mathfrak{p}\Big\{|0\rangle^{\otimes(n-1)}|d-1\rangle + \sum_{j_1+j_2=d-1} |0\rangle^{\otimes(n-2)}|j_1 j_2\rangle\Big\},$$

where $1 \leq j_1 \leq j_2 \leq d-2$. By a rotation one can write a change of basis $|j_1\rangle \to \frac{1}{\sqrt{2}}(|j_1\rangle + \mathrm{i}|j_2\rangle)$ and $|j_2\rangle \to \frac{1}{\sqrt{2}}(|j_1\rangle - \mathrm{i}|j_2\rangle)$ for $1 \leq j_1 \neq j_2 \leq d-2$ where $\mathrm{i} = \sqrt{-1}$. So we can rewrite the $n$-qudit M state as follows

$$|\,\mathrm{M}(d,n)\rangle = \sum_{\mathfrak{p}\in\mathfrak{S}_n} \mathfrak{p}\Big\{|0\rangle^{\otimes(n-1)}|d-1\rangle + \sum_{j=1}^{d-2} |0\rangle^{\otimes(n-2)}|jj\rangle\Big\}.$$

Now, we can write $n$-qudit M state as a limit of a $n$-qudit GHZ-equivalent state as follows

$$|\,\mathrm{M}(d,n)\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\sum_{j=1}^{d-2}(|0\rangle + \varepsilon|j\rangle + \frac{\varepsilon^2}{d-2}|d-1\rangle)^{\otimes n} - \frac{1}{\varepsilon}\big(|0\rangle + \varepsilon^2\sum_{j=1}^{d-2}|j\rangle\big)^{\otimes n}$$
$$- (d-2-\frac{1}{\varepsilon})|0\rangle^{\otimes n}\Big).$$

# Direct Sum

Lemma: $\text{rk}(\mathcal{G}(d,m) \boxtimes \mathcal{T}) = \text{rk}(\oplus^d \mathcal{T})$.

Theorem: Let $\mathcal{S} \in U_1 \otimes \cdots \otimes U_n$ be a tensor. If $\mathcal{T} \in V_1 \otimes \cdots \otimes V_n$ is a persistent tensor and $d_k = \dim V_k$, then

$$\text{rk}(\mathcal{S} \oplus \mathcal{T}) \geq \text{rk}(\mathcal{S}) + \sum_{k=1}^{n-1} d_k - n + 2\,.$$

Corollary: $\text{rk}(\mathcal{G}(d_1,m) \boxtimes \mathcal{M}(d_2,n)) = d_1\big((n-1)d_2 - n + 2\big)$.

Corollary: Concerning the following inequalities

$$\text{rk}(\mathcal{S} \boxtimes \mathcal{T}) \leq \text{rk}(\mathcal{S} \otimes \mathcal{T}) \leq \text{rk}(\mathcal{S})\,\text{rk}(\mathcal{T})\,.$$

we have $\text{rk}(\mathcal{G}(d_1,m) \otimes \mathcal{M}(d_2,n)) = d_1\big((n-1)d_2 - n + 2\big)$.

# Tensor Rank, Border Rank, SLOCC Transformation, & Degeneration

- It is known that a multiqudit GHZ-equivalent state can be transformed into a quantum state $|\psi\rangle$ iff $d \geq \mathrm{rk}(|\psi\rangle)$. So the tensor rank of a quantum state can be characterized as follows

$$\mathrm{rk}(|\psi\rangle) = \min \left\{ d \,\middle|\, |\,\mathrm{GHZ}(d,n)\rangle \xrightarrow{\mathrm{SLOCC}} |\psi\rangle \right\}.$$

- Similarly, the border rank of a quantum state $|\psi\rangle$ is the smallest $d$ such that a multiqudit GHZ-equivalent state degenerates into it, i.e.,

$$\mathrm{brk}(|\psi\rangle) = \min \left\{ d \,\middle|\, |\,\mathrm{GHZ}(d,n)\rangle \overset{\mathrm{SLOCC}}{-\!-\!\!\rightarrow} |\psi\rangle \right\}.$$

# Entanglement Transformation: Multiqudit GHZ and M States

Proposition: Let $|\psi\rangle \in V_1 \otimes \cdots \otimes V_n$ be an $n$-partite quantum state and let $\mathrm{rk}_S(\psi)$ denotes the Schmidt rank of $|\psi\rangle$ for any bipartite cut (bipartition) $S|\overline{S}$ where $S \subseteq [n]$ and $\overline{S} = [n] \setminus S$. For any bipartitions we have

$$\omega(\psi, \varphi) \geq \max_{S \subseteq [n]} \frac{\log \mathrm{rk}_S(\varphi)}{\log \mathrm{rk}_S(\psi)} .$$

Theorem: Let $|\psi\rangle$ and $|\varphi\rangle$ be two n-partite quantum states. If $|\psi\rangle$ degenerates into $|\varphi\rangle$ via SLOCC, then $\omega(\psi, \varphi) \leq 1$.

Theorem: An $n$-qudit GHZ-equivalent state can be transformed into an $n$-qudit M-equivalent state by asymptotic SLOCC with rate one, i.e.,

$$\omega(\mathcal{G}(d, n), \mathcal{M}(d, n)) = 1 .$$

**To the memory of**
**Mahsa Amini**
**a 22 yo girl who was killed**
**by morality police in Iran.**

# Thanks For Your Attention.