MIXED STATES AND ENTANGLEMENT

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ABSTRACT. This is an attempt at setting a mathematical framework for the interdisciplinary discussions taking place during the workshop "Tensors from the physics point of view" taking place in October 2022 at IMPAN Warsaw in the framework of the AGATES semester program. The main goal is to make a kind a "dictionary" between the two communities; and to agree on notation and terminology. Suggestions, corrections, questions and comments, especially pedantic ones, are more than welcome.

In this document, vectors in $\mathbb{C}^d$ will be denoted by $|\alpha\rangle$. All vector spaces are equipped with a Hermitian inner product (i.e. it is a finite-dimensional Hilbert space). This induces a conjugate-linear isomorphism

$$\begin{aligned} \mathbb{C}^d &\to (\mathbb{C}^d)^* \\ |\alpha\rangle &\mapsto \langle\alpha| \,. \end{aligned} \tag{0.1}$$

If you like to think about row and column vectors, the map above sends a column vector $\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}$ to the row vector $\begin{pmatrix} \overline{a_1} & \dots & \overline{a_d} \end{pmatrix}$.

1. Pure and mixed states

Definition 1.1. We work in a space $\mathcal{H} = \mathbb{C}^d$.

- A *pure state* is a nonzero vector $|\alpha\rangle \in \mathbb{C}^d$.
- A state is called *normed* if $\langle\alpha|\alpha\rangle = 1$.

In what follows, we will consider the space of operators $\mathrm{End}(\mathbb{C}^d) \cong (\mathbb{C}^d)^* \otimes \mathbb{C}^d$, which are just $d \times d$ matrices. Elements of this space can be written in the from $\sum_i c_i |\alpha\rangle \langle\beta|$. Note that we have a natural involution

$$(\mathbb{C}^d)^* \otimes \mathbb{C}^d \xrightarrow{(.)^H} (\mathbb{C}^d)^* \otimes \mathbb{C}^d$$

induced by $|\alpha\rangle \langle\beta| \to |\beta\rangle \langle\alpha|$. In matrix language, this sends a matrix to its conjugate transpose.

Definition 1.2. We still work in a space $\mathcal{H} = \mathbb{C}^d$.

- A *mixed state* or *density operator* is an element $\rho \in (\mathbb{C}^d)^* \otimes \mathbb{C}^d \cong \mathrm{End}(\mathbb{C}^d)$ which is Hermitian[1], positive-semidefinite, and has trace 1. In other words, $\rho \in (\mathbb{C}^d)^* \otimes \mathbb{C}^d$ is a mixed state if it can be written as

$$\rho = \sum_i \lambda_i |\alpha_i\rangle \langle\alpha_i|, \tag{1.1}$$

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[1] i.e. fixed under the involution we just mentioned

where $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$, $\lambda_i \in \mathbb{R}_{\geq 0}$, and $\sum \lambda_i = 1$.
- The *rank one projection operator* associated to a normed pure state $|\alpha\rangle$ is the element
$$|\alpha\rangle\langle\alpha| = |\alpha\rangle \otimes \langle\alpha| \in (\mathbb{C}^d)^* \otimes \mathbb{C}^d \cong \mathrm{End}(\mathbb{C}^d).$$
One easily checks that this is indeed a density operator.

Remark 1.3. Two normed pure states $|\alpha\rangle, |\beta\rangle$ give rise to the same projection operator if and only if they agree up a phase; i.e. $|\alpha\rangle = c|\beta\rangle$ where $c = e^{i\phi} \in \mathbb{C}$ has norm 1. Points in the projective space $\mathbb{P}(\mathbb{C}^d)$ correspond to states up to a phase, and we get an embedding $\mathbb{P}(\mathbb{C}^d) \hookrightarrow (\mathbb{C}^d)^* \otimes \mathbb{C}^d \cong \mathrm{End}(\mathbb{C}^d)$ whose image consists of the rank one density operators.

2. Entanglement for two qudits

Definition 2.1. We work in a space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$.
- A pure state $|\alpha\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is called *separable* if it is of the form $|\alpha\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$, where $|\alpha_i\rangle \in \mathbb{C}^{d_i}$.
- A mixed state
$$\rho \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes (\mathbb{C}^{d_1})^* \otimes (\mathbb{C}^{d_2})^* \cong \mathbb{C}^{d_1} \otimes (\mathbb{C}^{d_1})^* \otimes \mathbb{C}^{d_2} \otimes (\mathbb{C}^{d_2})^*$$
is called *simply seperable* if it is of the form $\rho_1 \otimes \rho_2$, where $\rho_i \in \mathbb{C}^{d_1} \otimes (\mathbb{C}^{d_1})^*$.
- A mixed state ρ is called *seperable* if it is a convex combination of simply seperable states, i.e. it is of the from
$$\rho = \sum_{k=1}^{N} p_k \rho^k, \tag{2.1}$$
where $\rho^k = \rho_1^k \otimes \rho_2^k$ are simply seperable states, $p_i \in \mathbb{R}_{\geq 0}$, and $\sum p_i = 1$.
- A (pure or mixed) state which is not seperable is called *entangled*.

Remark 2.2.
- In (2.1), one can without loss of generality assume that the ρ_i^k are rank one projection operators, i.e. the seperable mixed states are precisely the ones of the form
$$\rho = \sum_{k=1}^{N} p_k (|\alpha_1^k\rangle\langle\alpha_1^k|) \otimes (|\alpha_2^k\rangle\langle\alpha_2^k|) = \sum_{k=1}^{N} p_k |\alpha_1^k\alpha_2^k\rangle\langle\alpha_1^k\alpha_2^k|, \tag{2.2}$$
where $p_i \in \mathbb{R}_{\geq 0}$, and $\sum p_i = 1$.
- Our two definitions of "seperable" agree: it's trivial to check that if a pure state $|\alpha\rangle$ is seperable, then its projection operator is seperable (even simply seperable). On the other hand, if we have a seperable mixed state that comes from a pure state (i.e. it is a rank one operator), then there can be only one summand in (2.2), hence our state is indeed the projection operator of a pure seperable state.

3. Partial transpose and the PPT criterion

Coming soon...

4. Entanglement for many qudits

Coming soon...