Algebraic geometry and many body entanglement

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Symmetries in Physics

- A symmetry K of a space M – *an action of* K *on* M *that preserves some structure on* M

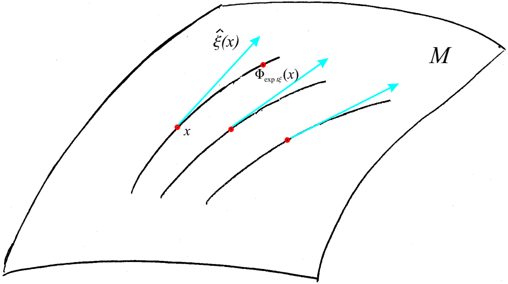
$$\Phi : K \times M \to M,$$
$$\Phi_g : M \to M,$$
$$\Phi_{g_1 g_2}(x) = \Phi_{g_1}(\Phi_{g_2}(x))$$

- (M, ω) – a symplectic manifold.
- ω nondegenerate ($\det \omega_{ij} \neq 0$) and closed ($d\omega = 0$).

Fundamental vector fields

- Symmetry of M: K – compact, connected semisimple Lie group, $\Phi_g^*\omega = \omega$ for all $g \in K$.
- $\mathfrak{k}$ – Lie algebra of K.
- For $\xi \in \mathfrak{k}$ the fundamental vector field $\hat{\xi}$:

$$\hat{\xi}(x) = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp t\xi}(x).$$



The momentum map

$$\mathcal{L}_{\hat{\xi}}\omega = 0, \quad i_{\hat{\xi}}d\omega + di_{\hat{\xi}}\omega = 0,$$
$$d\omega(\hat{\xi}, \cdot) = 0.$$

- For $\hat{\xi}$ there is $\mu_\xi : M \to \mathbb{R}$ s.t.:

$$d\mu_\xi(\cdot) = \omega(\hat{\xi}, \cdot)$$

- μ_ξ is a Hamilton function of $\hat{\xi}$
- Functions μ_ξ can be chosen s.t.

$$\mu_\xi(x) = (\mu(x)|\xi), \quad \mu(x) \in \mathfrak{k},$$
$$\mu\left(\Phi_g(x)\right) = \mathrm{Ad}_g\mu(x) = g\mu(x)g^{-1}.$$

where $(\cdot|\cdot)$ is a Ad_K-invariant inner product on $\mathfrak{k}$.

- The momentum map: $\mu : M \to \mathfrak{k}$

QM of qubits

- Qubit – 2-level quantum system, $\mathbb{C}^2 = \mathrm{Span}_{\mathbb{C}}\{|0\rangle, |1\rangle\}$
- L qubits: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_L$, where $\mathcal{H}_i \simeq \mathbb{C}^2$
- Pure states of L qubits - points in $M = \mathbb{P}(\mathcal{H})$
- Observables – hermitian operators on $\mathcal{H}$, i.e. $X \in i\mathfrak{u}(\mathcal{H})$
- $\frac{\langle\phi|X\phi\rangle}{\langle\phi|\phi\rangle}$ – expectation value of the observable X in the state $[\phi]$

$$\frac{\langle\phi|X\phi\rangle}{\langle\phi|\phi\rangle} = \mathrm{tr}\rho([\phi])X, \quad \rho([\phi]) \geq 0.$$

- $\rho([\phi])$ – density matrix, in our case $\rho([\phi]) = P_\phi$

Reduced density matrix

- Assume that only 1-qubit measurements are available
- $i\mathfrak{u}(2) \oplus \ldots \oplus i\mathfrak{u}(2)$ - 1-qubit observables for L qubits
- $X = (X_1, X_2, \ldots, X_L)$

$$\frac{\langle\phi|X\phi\rangle}{\langle\phi|\phi\rangle} = \sum_{k=1}^{L} \frac{\langle\phi|X_k\phi\rangle}{\langle\phi|\phi\rangle} = \sum_{k=1}^{L} \mathrm{tr}\rho_k([\phi])X_k$$

- $\rho_k([\phi]) \in i\mathfrak{u}(2)$ and $\rho_k([\phi]) \geq 0$.
- $\{\rho_1([\phi]), \rho_2([\phi]), \ldots, \rho_L(\phi)\}$ encode information about all one-qubit measurments

Local measurments and the momentum map

- $M = \mathbb{P}(\mathcal{H})$ – Kähler manifold with Fubini-Study ω_{FS}
- Natural action of $K = SU(2) \times \ldots \times SU(2)$ on M preserves ω_{FS}.
- $\mathfrak{k} = \mathfrak{su}(2) \oplus \ldots \oplus \mathfrak{su}(2)$ - Lie algebra of K
- The momentu map $\mu : \mathbb{P}(\mathcal{H}) \to \mathfrak{k}$:

$$(\mu([\phi])|X) = i\frac{\langle\phi|X\phi\rangle}{\langle\phi|\phi\rangle},\ \phi \in \mathcal{H},\ X \in \mathfrak{k}$$

- $\mu([\phi])$ encodes expectation values of all local measurements

$$\mu([\phi]) = i\left\{\rho_1([\phi]) - \frac{1}{2}I, \rho_2([\phi]) - \frac{1}{2}I, \ldots, \rho_L(\phi) - \frac{1}{2}I\right\}$$

Entanglement

- $\phi \in \mathcal{H}$ is not entangled iff it is a simple tensor
- Entanglement is preserved by Stochastic Local Operations and Classical Communication (SLOCC)
- $G = K^{\mathbb{C}} = SL(2,\mathbb{C})^{\times L}$ - invertible SLOOC

$$g.\phi = \frac{(g_1 \otimes \cdots \otimes g_L)\phi}{\|(g_1 \otimes \cdots \otimes g_L)\phi\|}, \ g_k \in SL(2,\mathbb{C}), \ \phi \in \mathcal{H}.$$

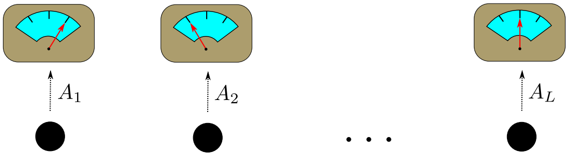
- Two states $[\phi_1]$ and $[\phi_2]$ are G-equivalent iff

$$[g.\phi_1] = [\phi_2], \, g \in G$$

- $\mathcal{C}_\phi := G.\phi = \{g.\phi : g \in G\}$ – a class of states with the entanglement type of $[\phi]$

Problems of SLOCC entanglement clasification

- Number of classes $\mathcal{C}_\phi$ is infinite starting from the system of four qubits.
- Number of parameters required to distinguish between classes C_ϕ grows exponentially with the number of qubits.
- These parameters, e.g. G-invariant polynomials typically lack physical meaning and are not measureable.
- We want to introduce a classification, which is more robust by organising classes $\mathcal{C}_\phi$ into a finite number of families that can be distinguished using single qubit measurements.



1-qubit RDMs

- $\rho_i([\phi])$ - the i-th one-qubit Reduced Density Matrix (RDM)

$$\mu([\phi]) = \left\{\rho_1([\phi]) - \frac{1}{2}I, \ldots, \rho_L([\phi]) - \frac{1}{2}I\right\}$$

- The ordered spectrum of $\rho_i([\phi]) - \frac{1}{2}I$ is given by

$$\sigma\left(\rho_i([\phi]) - \frac{1}{2}I\right) = (-\lambda_i, \lambda_i)\,,\ \ \lambda_i \in [0, \frac{1}{2}].$$

- The collection of spectra for $[\phi] \in \mathbb{P}(\mathcal{H})$:

$$\Psi : \mathbb{P}\mathcal{H} \to \left[0, \frac{1}{2}\right]^{\times L},\ \ \Psi([\phi]) = \{\lambda_1, \lambda_2, \ldots, \lambda_L\}.$$

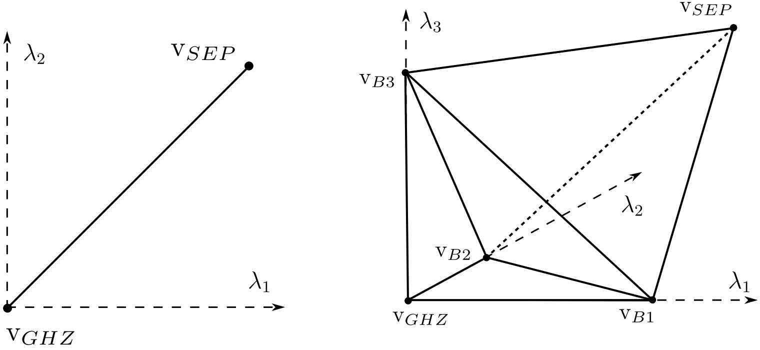
First Convexity Theorem

- $\Delta_{\mathcal{H}} := \Psi(\mathbb{P}\mathcal{H})$ is a convex polytope.
- Follows from the momentum map convexity theorem (Kirwan '84)
- Higuchi, Sudbery, Szulc '03 - This polytope is given by the intersection of

$$\forall_i \ \left(\frac{1}{2} - \lambda_i\right) \leq \sum_{j \neq i} \left(\frac{1}{2} - \lambda_j\right),$$

with the cube $\left[0, \frac{1}{2}\right]^{\times L}$.

$\Delta_{\mathcal{H}}$ for 2 and 3 qubits



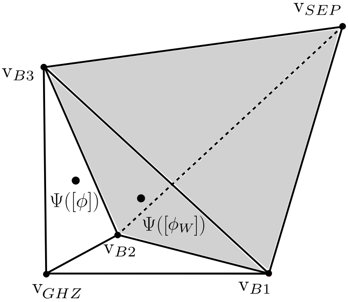
(a) (b)

Second Convexity Theorem

- $\mathcal{C}_\phi = [G.\phi]$
- $\Delta_{\mathcal{C}_\phi} = \Psi(\overline{\mathcal{C}_\phi})$ is a convex polytope.
- Follows from the convexity theorem of Brion '87
- $\Delta_{\mathcal{C}_\phi}$ is called an Entanglement Polytope (EP)
- Introduced to QI in '12 (AS, Oszmaniec, Kuś) and (Walter, Doran, Gross, Christandl)
- Although for $L \geq 4$ the number of classes $\mathcal{C}$ is infinite, the number of polytopes $\Delta_{\mathcal{C}}$ is always finite!
- Brion's theorem: Finding EPs requires knowing the generating set of covariants.
- This was solved only up to 4 qubits (Briand, Luque, J.-Y. Thibon 2003).

Properties of Entanglement Polytopes

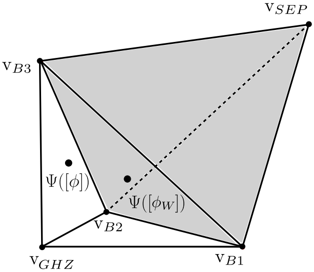
- We define equivalence relation: $\mathcal{C}_\phi \sim \mathcal{C}_\psi$ iff $\Delta_{\mathcal{C}_\phi} = \Delta_{\mathcal{C}_\psi}$
- Entanglement polytopes are typically not disjoint, $\Delta_{\mathcal{C}} \cap \Delta_{\mathcal{C}'} \neq \emptyset$.
- Example: $\Delta_{\mathcal{C}_{GHZ}} = \Delta_{\mathcal{H}}$ thus $\Delta_{\mathcal{C}_\phi} \subset \Delta_{\mathcal{C}_{GHZ}}$ for every C_ϕ
- Entanglement polytopes can be regarded as entanglement witnesses.



$$|\phi_W\rangle = \frac{1}{\sqrt{3}}\left(|011\rangle + |101\rangle + |110\rangle\right)$$

$$|\phi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

Properties of Entanglement Polytopes



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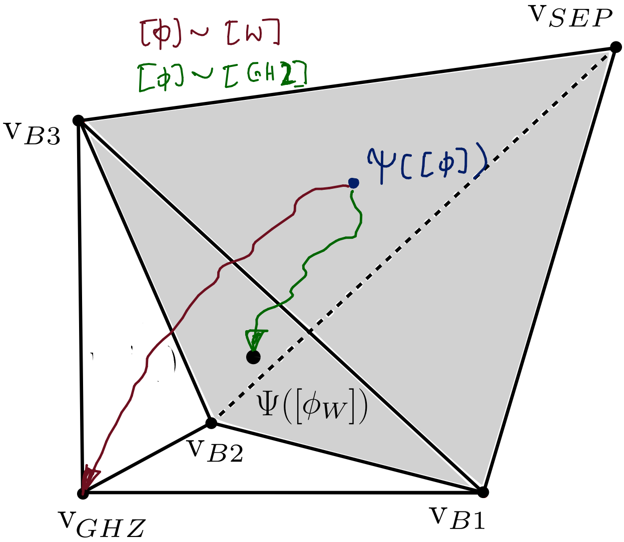
EPs as entanglement witnesses:

- For $[\phi] \in \mathbb{P}(\mathcal{H})$ we give a list of polytopes that do not contain $\Psi([\phi])$.
- The decision-making power of EPs is determined by the volume of the region in $\Delta_{\mathcal{H}}$ where many EPs overlap.
- Problem: Decision-making power of EPs for large L.

Critical points

- Finding entanglement polytopes, even for five qubits, is in fact intractable!
- We want to partially characterise EPs without finding them.
- For a polytope $\Delta_{\mathcal{C}}$ let $\overline{\lambda}_{\mathcal{C}}$ be the point that is closest to the origin $\overline{0}$.
- $\|\overline{\lambda}_{\mathcal{C}}\|^2 = \min_{[\phi]\in\mathcal{C}} \|\Psi(\phi)\|^2$
- For any $[\phi] \in \mathbb{P}(\mathcal{H})$ there is a sequence of SLOCC operations that transforms $[\phi]$ into $[\phi']$, such that $\Psi([\phi']) = \overline{\lambda}_{\mathcal{C}}$.
- We define equivalence relation: $\mathcal{C}_\phi \sim \mathcal{C}_\psi$ iff $\overline{\lambda}_{\mathcal{C}_\phi} = \overline{\lambda}_{\mathcal{C}_\psi}$
- Our aim is to understand the distribution of $\|\overline{\lambda}_{\mathcal{C}}\|^2$ in $\Delta_{\mathcal{H}}$ for large number of qubits.

Example

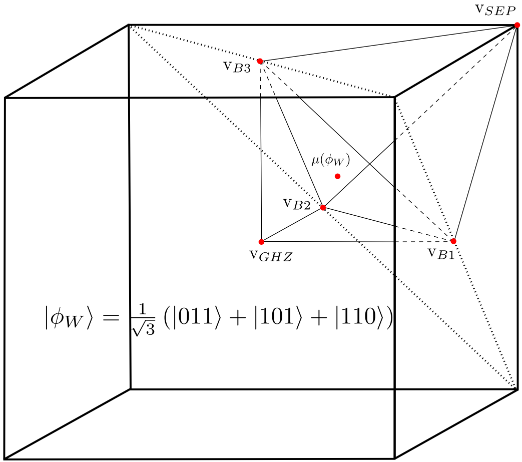


Procedure for finding $\overline{\lambda}_C$ for L qubits

- '15 TM and A. Sawicki - the procedure for finding $\overline{\lambda}_C$ using momentum map results of Kirwan and Ness

1. Construct L-dimensional hypercube whose vertices have coordinates $\pm\frac{1}{2}$.
2. Chose L out of 2^L vertices and consider the plane P containing the chosen points .
3. Find the closest point p to the origin $\overline{0}$ in P.
4. Point $p = \overline{\lambda}_{\mathcal{C}}$ for some $\Delta_{\mathcal{C}}$ iff p does not lie on an edge of the hypercube.

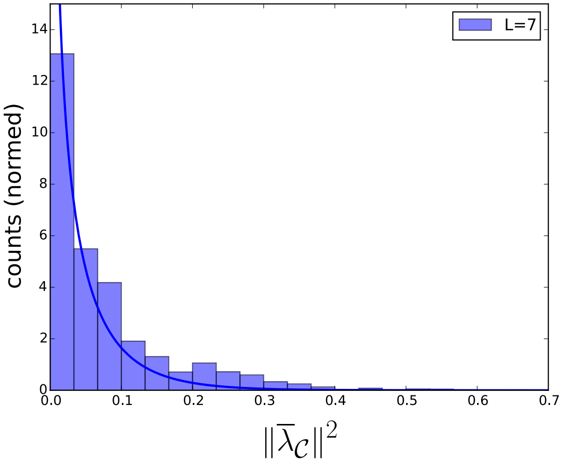
3 qubits



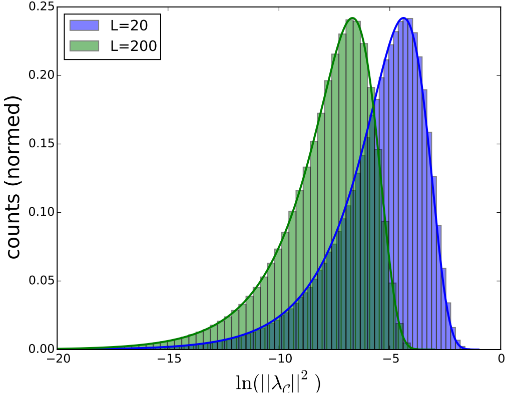
$$|\phi_W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle), \ |\phi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|\phi_{B1}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes (|11\rangle + |00\rangle))$$

Histogram for 7 qubits vs. $Gamma\left(\frac{1}{2}, 2L\right)$



Histograms for 20 and 200 qubits, sample of 10^6 points



$$\|\overline{\lambda}_C\|^2 \sim Gamma\left(\frac{1}{2}, 2L\right), \ \mathbb{E}\left[\|\overline{\lambda}_C\|^2\right] = \frac{1}{4L}.$$

$$Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

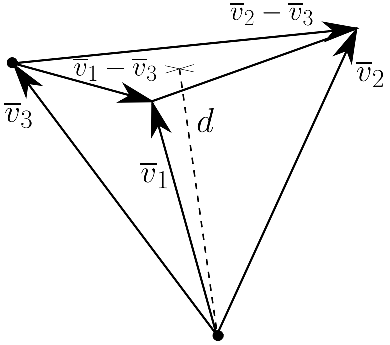
Procedure for finding $\|\overline{\lambda}_{\mathcal{C}}\|^2$ for L qubits

- For vectors $\overline{v}_1, \dots, \overline{v}_k \in \mathbb{R}^n$ let

$$G(\overline{v}_1, \dots, \overline{v}_k) = \begin{pmatrix} (\overline{v}_1|\overline{v}_1) & \dots & (\overline{v}_1|\overline{v}_k) \\ \vdots & \ddots & \dots \\ (\overline{v}_k|\overline{v}_1) & \dots & (\overline{v}_k|\overline{v}_k) \end{pmatrix}$$

- $|G(\overline{v}_1, \dots, \overline{v}_k)| := \det G(\overline{v}_1, \dots, \overline{v}_k)$

Example



$$V = \frac{1}{3} S d, \quad d^2 = \frac{3^2 V^2}{S^2}$$

$$V^2 = \frac{1}{3!} |G(\overline{v}_1, \overline{v}_2, \overline{v}_3)| \qquad S^2 = \frac{1}{2!} |G(\overline{v}_1 - \overline{v}_3, \overline{v}_2 - \overline{v}_3)|$$

$$d^2 = \frac{|G(\overline{v}_1, \overline{v}_2, \overline{v}_3)|}{|G(\overline{v}_1 - \overline{v}_3, \overline{v}_2 - \overline{v}_3)|},$$

Formula for $\|\overline{\lambda}_{\mathcal{C}}\|^2$

$$\|\overline{\lambda}_{\mathcal{C}}\|^2 = \frac{1}{4}\frac{|G(\overline{v}_1, \ldots, \overline{v}_L)|}{|G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L)|},$$

where $\overline{v}_i \in \mathbb{R}^L$ are vectors with ± 1 entries – Bernoulli vectors

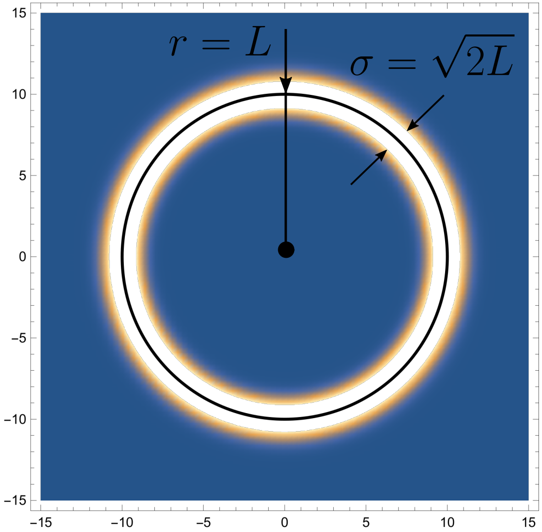
The model

- Vertices of the L-dimensional cube with Bernoulli vertices are uniformly distributed on S^{L-1} with $r^2 = L$.
- Let $\overline{v} = (v_1, \ldots, v_L)^t \in \mathbb{R}^L$ be a Gaussian vector, i.e. $v_i \sim N(0,1)$.

$$\overline{v} \sim \frac{\exp\left(-\frac{1}{2}\|v\|^2\right)}{\sqrt{(2\pi)^L}}$$

- The distribution of $\overline{v}$ is isotropic. $\|\overline{v}\|^2$ is χ^2_L with the mean L and $\sigma = \sqrt{2L}$
- When $L \to \infty$ the ratio $\frac{\sqrt{2L}}{L} \to 0$
- **Problem**: Calculate distribution of $\frac{|G(\overline{v}_1,\ldots,\overline{v}_L)|}{|G(\overline{v}_1-\overline{v}_L,\ldots,\overline{v}_{L-1}-\overline{v}_L)|}$ for $\overline{v}_i \sim N(\overline{0}, I)$

The model



Warm up

$$\frac{|G(\overline{v}_1,\ldots,\overline{v}_L)|}{|G(\overline{v}_1,\ldots,\overline{v}_{L-1})|}$$

- Let $\overline{v}_i \sim N(\overline{0}, I)$ and consider $G := G(\overline{v}_1,\ldots,\overline{v}_L)$.
- G is a positive symmetric matrix distributed according to Wishart distribution $\mathcal{W}(L, I)$

$$\frac{1}{2^{L^2/2}\Gamma_L(\frac{L}{2})}|G|^{-\frac{1}{2}}e^{-(1/2)\mathrm{tr}G}$$

- Cholesky decomposition: $G(\overline{v}_1,\ldots,\overline{v}_L) = TT^t$, where T-lower triangular matrix
- Theorem (Bartlett) T_{ii}^2 are independent random variables

$$T_{ii}^2 \sim Gamma\left(\frac{L-i+1}{2}, \frac{1}{2}\right)$$

Warm up

- We are interested in T_{LL}^2

$$T_{LL}^2 = \frac{|G(\overline{v}_1, \dots, \overline{v}_L)|}{|G(\overline{v}_1, \dots, \overline{v}_{L-1})|} = \frac{|G|}{|G^{[L,L]}|} \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

- $|G^{[L,L]}|$ is the (L, L) minor of G.

Finding the distribution of $\|\overline{\lambda}_{\mathcal{C}}\|^2$

- Using antisymmetry of the determinant

$$|G(\overline{v}_1, \ldots, \overline{v}_{L-1}, \overline{v}_L)| = |G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L, \overline{v}_L)|$$

- Let $G' := G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L, \overline{v}_L)$
- Thus

$$\frac{|G(\overline{v}_1, \ldots, \overline{v}_L)|}{|G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L)|} = \frac{|G'|}{|G'^{[L,L]}|}$$

- What is the distribution of G'

Finding the distribution of $\|\overline{\lambda}_{\mathcal{C}}\|^2$

$$G' = A^t G A$$

- A lower triangular matrix

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ -1 & -1 & \dots & -1 & -1 \end{pmatrix}$$

- Theorem: Assume $G \sim \mathcal{W}(L, I)$ then $A^t G A \sim \mathcal{W}(L, \Sigma)$, where $\Sigma := A^t A$
- G' is distributed according to the Wishart distribution $\mathcal{W}(L, \Sigma)$

Finding distribution of $\|\overline{\lambda}_{\mathcal{C}}\|^2$

- $\mathcal{W}(L, \Sigma)$ is a Gramm matrix of vectors $\overline{w}_i \sim N(0, \Sigma)$
- Conclusion: For vectors $\overline{v}_i \sim N(0, I)$ and vectors $\overline{w}_i \sim N(0, \Sigma)$ the distribution of

$$\frac{|G(\overline{v}_1, \ldots, \overline{v}_L)|}{|G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L)} \text{ and } \frac{|G(\overline{w}_1, \ldots, \overline{w}_L)|}{|G(\overline{w}_1, \ldots, \overline{w}_{L-1})|}$$

  are the same.
- We need to calculate distribution of $\frac{|G(\overline{w}_1, \ldots, \overline{w}_L)|}{|G(\overline{w}_1, \ldots, \overline{w}_{L-1})|}$ for $\overline{w}_i \sim N(0, \Sigma)$

Finding distribution of $\|\overline{\lambda}_{\mathcal{C}}\|^2$

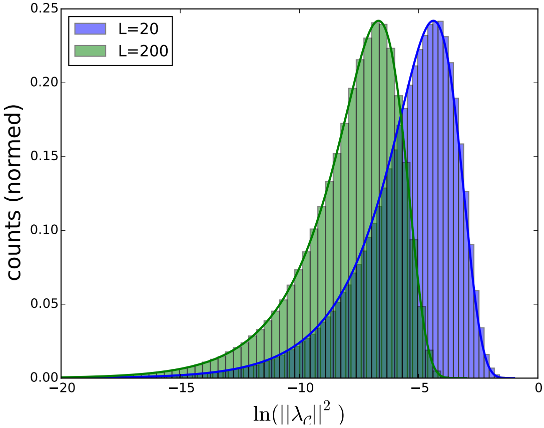
- Let RR^t be the Cholesky decomposition of $\Sigma = A^t A$,
- Let TT^t be the Cholesky decomposition of $G \sim \mathcal{W}(L, I)$
- $RTT^t R^t \sim \mathcal{W}(L, \Sigma)$

$$\frac{|G(\overline{w}_1, \ldots, \overline{w}_L)|}{|G(\overline{w}_1, \ldots, \overline{w}_{L-1})|} = (RT)^2_{LL} = R^2_{LL} T^2_{LL} = \frac{1}{L} T^2_{LL}$$

- $T^2_{LL} \sim Gamma(\frac{1}{2}, \frac{1}{2})$

$$|\overline{\lambda}_{\mathcal{C}}|^2 \sim Gamma\left(\frac{1}{2}, 2L\right)$$

Histograms for 20 and 200 qubits, sample of 10^6 points



$$\|\overline{\lambda}_C\|^2 \sim Gamma\left(\frac{1}{2}, 2L\right), \ \mathbb{E}\left[\|\overline{\lambda}_C\|^2\right] = \frac{1}{4L}.$$

Papers

1. O. Słowik, M. Hebenstreit, B. Kraus, and AS, *A link between symmetries of critical states and the structure of SLOCC classes in multipartite systems*, Quantum 4 300, (2020)
2. T. Maciążek, A.S., *Asymptotic properties of entanglement polytopes for large number of qubits*, J. Phys. A: Math. Theor, 51, 7, 07LT01, (2018)
3. T. Maciążek, A.S., *Critical points of the linear entropy for pure L-qubit states*, J. Phys. A: Math. Theor. 48 045305, (2015)
4. AS, M. Oszmaniec, M. Kuś, *Convexity of momentum map, Morse index, and quantum entanglement*, Reviews in Mathematical Physics, 26, 1450004, (2014)
5. AS, M. Oszmaniec, M. Kuś, *Critical sets of the total variance can detect all stochastic local operations and classical communication classes of multiparticle entanglement*, Phys. Rev. A 86, 040304(R) (2012)