Measures of quantum entanglement

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Tensors from the physics viewpoint
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Overview

- Notations and facts
- Geometric measure of entanglement
- Ranks of tensors and identifiability
- Symmetric and skew-symmetric tensors
- Spectral and nuclear norms
- Nuclear rank

Notations and facts I

A pure state in QI correspond to a vector of unit length in $\mathcal{H}$: $\mathbf{x} = |x\rangle$

a Hilbert space over $\mathbb{C}$ inner product $\langle \mathbf{y}, \mathbf{x}\rangle = \mathbf{y}^*\mathbf{x} = \langle y|x\rangle$, norm $\|\mathbf{x}\|$

To a system consisting of d-parts correspond d-tensor product

$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d$, for simplicity $\mathcal{H}_i = \mathbb{C}^{n_i}$

$\mathbf{v} \in \mathcal{H}$ is unentangled if it is a product state

$\mathbf{v} = \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d = |\mathbf{v}_1\rangle \cdots |\mathbf{v}_d\rangle = |i_1 \cdots i_d\rangle$

Notations and facts II

$[n] := \{1, \ldots, n\}$, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$,$[\mathbf{n}] = [n_1] \times \cdots \times [n_d]$

$\mathbb{F}$ a field, in QI $\mathbb{F} = \mathbb{C}$, $\otimes_{j=1}^d \mathbb{F}^{n_j} = \mathbb{F}^{n_1 \times \ldots \times n_d} = \mathbb{F}^{\mathbf{n}}$

d-mode tensor $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{n_1 \times \ldots \times n_d}$, $i_j \in [n_j], j \in [d]$

$d = 1$ vector: $\mathbf{x} = (x_1, \ldots, x_n)^\top$; $d = 2$ matrix $A = [a_{ij}]$, $d \geq 3$ tensor

rank one tensor $\mathcal{T} = [x_{i_1,1} x_{i_2,2} \cdots x_{i_d,d}] = \mathbf{x}_1 \otimes \mathbf{x}_2 \cdots \otimes \mathbf{x}_d = \otimes_{j=1}^d \mathbf{x}_j \neq 0$

IP: $\langle \mathcal{T}, \mathcal{S} \rangle = \sum_{i_j \in [n_j], j \in [d]} \overline{t_{i_1,\ldots,i_d}} s_{i_1,\ldots,i_d}$, $\quad \|\mathcal{T}\|_{HS} := \sqrt{\langle \mathcal{T}, \mathcal{T} \rangle}$

Contraction: $\mathcal{T} \times (\otimes_{j=2}^d \mathbf{x}_j) = \sum_{i_j \in [n_j], j \in \{2,\ldots,d\}} t_{i_1,\ldots,i_d} x_{i_j,j} \in \mathbb{F}^{n_1}$

$\mathcal{T} \times (\otimes_{j=1}^d \mathbf{x}_j) = \sum_{i_j \in [n_j], j \in \{1,\ldots,d\}} t_{i_1,\ldots,i_d} x_{i_j,j} \in \mathbb{F}$

$\mathrm{S}(n, \mathbb{F}) := \{\mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| = 1\}$ - unit sphere in $\mathbb{F}^n$, $(\mathbb{F} = \mathbb{C}, \mathbb{R})$

Notations and facts III

rank of tensor: $\operatorname{rank} \mathcal{T}$– minimal decomposition $\mathcal{T} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i}$

The computation of $\operatorname{rank} \mathcal{T}$ NP-complete over finite fields

NP-hard over $\mathbb{R}, \mathbb{C}$ for $d \geq 3$ - Håstad 1990

Short explanation why: let $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{n}}$

Denote $\mathcal{T}_k = [t_{i_1,\ldots,i_{d-1},k}] \in \otimes_{j=1}^{d-1} \mathbb{F}^{n_j}, k \in [n_d]$

$\operatorname{rank} \mathcal{T}$ - minimal number of rank-one tensors spanning

$\operatorname{span}(\mathcal{T}_1, \ldots, \mathcal{T}_{n_d}) \subset \otimes_{j=1}^{d-1} \mathbb{F}^{n_j}$

Notations and facts IV

unfolding tensor in mode: 1 $T_1(\mathcal{T}) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2\times\cdots\times n_d}$

equivalent to view $\mathcal{T}$ as bipartite state $\{i\} \cup \{[d] \setminus \{i\}\}$.

unfolded rank: $\mathrm{rank}_i\mathcal{T} = \mathrm{rank}\ T_i(\mathcal{T})$

$\max\{\mathrm{rank}_i T, i \in [d]\} \le \mathrm{rank}\ \mathcal{T}$

By choosing a basis in $\mathbb{F}^{n_i}$ containing column space of $T_i(\mathcal{T})$

we can assume $n_i = \mathrm{rank}_i\mathcal{T}$ for $i \in [d]$

Tensors and Quanta

State $\mathbf{x} \in \mathbb{C}^n$, $\|\mathbf{x}\| = 1$ (normalized, nonnormalized: $\mathbf{x} \neq \mathbf{0}$)

$\mathcal{T} \in \mathbb{C}^{n_1 \times \cdots \times n_d}$ d-partite state, shared by d parties

$T \in \mathbb{C}^{n_1 \times n_2}$ bipartite state, shared by Alice and Bob

$\mathcal{T}$ unentagled if $\mathrm{rank}\, \mathcal{T} = 1$, otherwise $\mathcal{T}$ entangled

$\mathrm{rank}\, \mathcal{T}$ measurement of entanglement

$\mathrm{rank}\, \mathcal{T} = 1 \iff \mathrm{rank}_i(\mathcal{T}) = 1$ for $i \in [d]$

Entanglement easy to detect

Geometric measure of entanglement

$\mathbf{P}(\mathbf{n}) = \{\otimes_{j=1}^{d} \mathbf{x}_j \in \mathbb{F}^{\mathbf{n}}, \|\mathbf{x}_j\| = 1\}$ unentangled states

$\|\mathcal{T}\|_{spec} = \max\{|\langle \otimes_{j=1}^{d} \mathbf{x}_j, \mathcal{T}\rangle|, \otimes_{j=1}^{d} \mathbf{x}_j \in \mathbf{P}(n)\} = \max\{\Re\langle \otimes_{j=1}^{d} \mathbf{x}_j, \mathcal{T}\rangle\}$

$\mathrm{gme}(\mathcal{T}) = \min\{\|\mathcal{T} - \otimes_{j=1}^{d} \mathbf{x}_j\|, \otimes_{j=1}^{d} \mathbf{x}_j \in \mathbf{P}(\mathbf{n})\} = \sqrt{2(1 - \|\mathcal{T}\|_{spec}})$

Maximally entangled state $\mathcal{T}$: $\|\mathcal{T}\|_{\mathrm{spec}}$ - minimal

$T \in \mathbb{F}^{m \times n}$, $\|T\|_{spec} = \sigma_1(A)$ - maximum singular value, spectral norm

$\|T\|^2 = \sum_{i=1}^{\min(m,n)} \sigma_i^2(A) \Rightarrow \|T\|_{spec} \geq \frac{1}{\sqrt{\min(m,n)}}$

Equality holds for Bell state: $T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

The matrix case

$T \in \mathbb{C}^{m\times n}, \|T\| = 1 \Rightarrow \mathrm{rank}\ T \in [\min(m,n)]$

$\mathrm{rank}\ T = 1$ iff T unentangled,

maximally entangled $\mathrm{rank}\ T = \min(m,n)$, and

$\sigma_i(T) = \frac{1}{\sqrt{\min(m,n)}}$ for $i \in [\min(m,n)]$.

Computation of tensor rank I

$\mathcal{T} \in \otimes_{j=1}^{d} \mathbb{C}^{n_j}, \quad 3 \le d, \quad 2 \le n_1 \le \cdots \le n_d$

Claim: $\mathrm{rank}\, \mathcal{T} > r$ iff the system of polynomial equations

$\sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i} - \mathcal{T} = 0$ is unsolvable

Hilbert nullstellensatz: the system of polynomial equations

$f_l(\mathbf{z}) = 0, \mathbf{z} \in \mathbb{C}^M, l \in [s]$ is not solvable iff (1) $\sum_{l=1}^{s} a_l(\mathbf{z}) f_l(\mathbf{z}) = 1$

Efficient HN: Let $\deg f_1 \ge \cdots \ge \deg f_s$. If (1) solvable over $\mathbb{C}[\mathbf{z}]$ then

(1) is solvable with (2) $\deg a_i f_i \le 2 \deg f_s \prod_{i=1}^{\min(M,s)-1} \deg f_i$ for $i \in [s]$

If $\deg f_s \ge 3$ one can drop factor 2 in (2)

Computation of tensor rank II

$\mathbf{z} = (\mathbf{x}_{1,1}, \ldots \mathbf{x}_{d,r})$, $M = M(r, \mathbf{n}) = r \sum_{j=1}^{d} n_j$, $s = N(\mathbf{n}) = \prod_{j=1}^{d} n_j$

Write $a_l(\mathbf{z})$ as a polynomial with unknown coefficients

(1) is system of linear equations with less $eN(\mathbf{n})d^{(M(r,\mathbf{n})-1)M(r,\mathbf{n})}$

variables, and $ed^{(M(r,\mathbf{n})-1)M(r,\mathbf{n})}$ equations

Gauss elimination complexity is $O(d^{3M(r,\mathbf{n})(M(r,n)-1)})$

Computation of tensor rank III

Algorithm for finding $\mathrm{rank}\,\mathcal{T}$:

Set $r = 1$, check if $\mathrm{rank}_i\,\mathcal{T} = 1$ for $i \in [d]$

if yes $\mathrm{rank}\,\mathcal{T} = 1$, finish

set $r = r + 1$, check if (1) solvable

if solvable, set $r = r + 1$,

if not $\mathrm{rank}\,\mathcal{T} = r$, finish

First identifiability result: Kruskal's theorem

$K(S)$-Kruskal rank of a finite set $S \subset \mathbb{F}^n \setminus \{\mathbf{0}\}$ is the maximum $\ell(\leq |S|)$:

every ℓ vectors in S are linearly independent

(1) $\mathcal{T} = \sum_{i=1}^r \otimes_{j=1}^d \mathbf{x}_{j,i} \in \otimes_{j=1}^d \mathbb{F}^{n_j}$, $d \geq 3$, $S_j = \{\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,r}\}$, $j \in [d]$

Kruskal's thm: If $2r + d - 1 \leq \sum_{j=1}^d K(S_j) \Rightarrow$

(1) unique rank decomposition (up to permuting the rank-one factors)

Kruskal 1976 $d = 3$ and $\mathbb{F} = \mathbb{R}$

Sidoropoulos-Bro 2000 $d \geq 4$ and $\mathbb{F} = \mathbb{R}$

Rhodes 2010 $d = 3$ and $\mathbb{F}$ any field $\Rightarrow d \geq 3$

More: Domanov-De Lathauwer, Chiantini-Ottaviani-Vannieuwenhoven

d-rank tensors and border rank

$V(r,\mathbf{n}) = \{\mathcal{T} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i} \in \otimes_{j=1}^{d} \mathbb{C}^{n_j}\}$-constructible set

(quasi-affine variety): a boolean combination of affine varieties.

$r_{\max}(d,\mathbf{n})$-the smallest $r > 1$ s.t. $V(r-1,\mathbf{n}) \subset V(r,\mathbf{n})$

Tensors of rank-r - $R(r,\mathbf{n}) = V(r,\mathbf{n}) \setminus V(r-1,\mathbf{n})$ is constructible

$V(1,\mathbf{n})$ an irreducible variety - Segre variety

$V(2,\mathbf{n})$ is not a closed set for $d \geq 3, \min\{n_j, j \in [d]\} \geq 2$:

$\mathcal{W}_d = \mathbf{x} \otimes^{d-1} \mathbf{y} + \mathbf{y} \otimes \mathbf{x} \otimes^{d-2} \mathbf{y} + \cdots + \otimes^{d-2}\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{y} + \otimes^{d-1}\mathbf{y} \otimes \mathbf{x}$

$\mathrm{rank}\ \mathcal{W}_d = d$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ linearly independent,

$\mathcal{W}_d = \lim_{t\to 0} t^{-1}((\mathbf{y} + t\mathbf{x})^{\otimes d} - \mathbf{y}^{\otimes d}) \Rightarrow \mathcal{W}_d \in \mathrm{Closure}(V(2,\mathbf{n}))$

border rank-$\mathrm{brank}\mathcal{T}$-the minimum r s.t. $\mathcal{T}$ is limit of tensors of rank-r

Bini-Lotti-Romani 1980

Generic rank and subgeneric rank conjecture

generic rank-$\mathrm{grank}(\mathbf{n})$ -the smallest $r > 1$ s.t. Closure $V(r, \mathbf{n}) = \mathbb{C}^{\mathbf{n}}$

generic rank $\mathrm{grank}(\mathbf{n})$-rank of a randomly chosen state in $\mathbb{C}^{\mathbf{n}}$

subgeneric rank $r < \mathrm{grank}(\mathbf{n})$

Conjecture: $\mathcal{T} \in V(r, \mathbf{n}), r < \mathrm{grank}(\mathbf{n})$ in critical range and

in general position is in $R(r, \mathbf{n})$, and has unique rank decomposition

partial results Chiantini-Ottaviani-Vannieuwenhoven 2014, 2017

Kruskal's theorem: If $2r + d - 1 \leq \sum_{j=1}^{d} \min(n_j, r)$ every tensor

$\mathcal{T} \in V(r, d)$ s.t. each $\{\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,r}\}$ in general linear position

is in $R(r, \mathbf{n})$ and has unique rank decomposition

Computation of generic rank

Assume $2 \le n_1 \le n_2 \le \cdots \le n_d$ (Recall that $\mathrm{rank}\, \mathcal{T} \le \prod_{j=1}^{d-1} n_j$)

Dimension count: $\mathrm{grank}(\mathbf{n}) \ge \mathrm{grank}_0(\mathbf{n}) := \lceil \frac{\prod_{j=1}^{d} n_i}{(\sum_{j=1}^{n} n_j)-d+1} \rceil$

(Number of parameters in $\otimes_{j=1}^{d} \mathbf{x}_j$ is $(\sum_{j=1}^{n} n_j) - d + 1$)

Terracini's lemma: $\mathbf{F}_r : (\mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_d})^r \to \mathbb{F}^{n_1 \times \cdots \times n_d}$

$\mathbf{F}_r = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i}$

$\mathrm{grank}(\mathbf{n})$ minimum r s. t. $\mathrm{rank}\, D(\mathbf{F}_r) = N(\mathbf{n})$ at a generic point

For $1 + \prod_{j=1}^{d-1}(n_j - 1) \le n_d \le \prod_{j=1}^{d-1} n_j$ we have $\mathrm{grank}(\mathbf{n}) = n_d$

Critical range: $2 \le n_1 \le n_2 \le \cdots \le n_d \le \prod_{j=1}^{d-1}(n_j - 1)$

Table of generic ranks: $\mathbf{n} = n^{\times d} := (n, \ldots, n) \in \mathbb{N}^d$

$d \setminus n$	2	3	4	5	6	7	8	9	
2	[1,6]2	3	4	5	6	7	8	9	
3	[6,7,8]2	[7,8]5	[7,8]7	[1,7,8]10	[7,8]14	[7,8]19	[1,7,8]24	[7]30	
4	[1,6,8]4	[8]9	[1,8]20	[8]37	[8]62	[8]97	**142**	**199**	
5	[1,6]6	**23**	[1,6]64	**149**	[1]300	**543**	.	.	
6	[1,6]10	[1]57	[1]216	[1,6]625	[1]1506	.	.	.	
7	[1,6]16	**146**	**745**	.	[1]6^5	.	[1]41944	.	
8	[6]29	**386**	.	.	.	[1,6]7^6	.	.	
9	[1,6]52	**1036**	.	.	.	.	[1,6]8^7	.	
10	[1,6]94	.	.	.	[1]1185612	.	.	[1,6]9^8	[1]1
11	[6]171	.	.	[1]1085070	.	.	.	.	
12	[1,6]316	[1]21258	.	.	.	.	.	.	
13	[1,6]586	[1,6]3^{10}	.	[1]23032135	.	.	.	.	
14	[6]1093	.	.	.	[1]1103720622	.	.	.	
15	[6]2048	.	.	.	.	.	.	.	
16	[1,6]3856	.	.	[1]2347506010	[1]$2^{16}3^{12}$	.	[1]2490928997440	.	[1]689655 (0.1)

Combinatorial upper bound on generic ranks

Conjecture: In critical range $\mathrm{grank}(\mathbf{n}) = \mathrm{grank}_0(\mathbf{n})$

with exceptional cases $\mathrm{grank}(\mathbf{n}) = \mathrm{grank}_0(\mathbf{n}) + 1$

Not known for most (n_1, n_2, n_3), but known for some $n^{\times d}$: all $2^{\times d}$

Let $\mathrm{dist}(\cdot, \cdot)$ be Hamming metric on $[\mathbf{n}] = [n_1] \times \cdots \times [n_d]$

$\mathcal{A} \subset [\mathbf{n}]$ dominating: $\mathrm{dist}(c, \mathcal{A}) \leq 1$ for all $c \in [\mathbf{n}] : |\mathcal{A}| \geq \mathrm{grank}_0(\mathbf{n})$

$\mathcal{B} \subset [\mathbf{n}]$ 3-separated: $\mathrm{dist}(c, d) \geq 3, c \neq d \in \mathcal{B}$: $|\mathcal{B}| \leq \mathrm{grank}_0(\mathbf{n})$

$\gamma(\mathbf{n}) \geq \kappa(\mathbf{n})$ min and max of dominating and 3-separated sets, respec.

Catalisano-Geramita-Gimigliano 2002: $\gamma(\mathbf{n}) \geq \mathrm{grank}(\mathbf{n})$

Proof: column space of $D(\otimes_{j=1}^{d} \mathbf{e}_{i_j})$ spanned by

$\otimes_{j=1}^{d} \mathbf{e}_{k_j}$- $\mathrm{dist}((k_1, \ldots, k_d), (i_1, \ldots, i_d)) \leq 1$

Combinatorial upper bound on generic ranks II

1-perfect code $\mathcal{C} \subset [\mathbf{n}]$: dominating and 3-separated ($|\mathcal{C}| = \mathrm{grank}_0(\mathbf{n})$)

If $[\mathbf{n}]$ has 1-perfect code then $\mathrm{grank}(\mathbf{n}) = \mathrm{grank}_0(\mathbf{n})$

perfect codes exist for $n_1 = \cdots = n_d = n = q^l, d = \frac{n^{a+1}-1}{n-1}$ q prime

$G = ([\mathbf{n}], E(\mathbf{n})), \{a, b\} \in E(\mathbf{n}) \iff \mathrm{dist}(a, b) = 1$

G-$M(\mathbf{n}) = \sum_{j=1}^{d}(n_j - 1)$-regular, $|[\mathbf{n}]| = N(\mathbf{n}) = \prod_{j=1}^{d} n_j$

Greedy algorithm: $\gamma(\mathbf{n}) \leq \mathrm{grank}_0(\mathbf{n}) O(\log M(\mathbf{n}))$

$\gamma(n^{\times d}) \leq \lceil \frac{n^d}{d(n-1)+1} \rceil O(d \log n)$

Can this upper bound improved for this special graph?

Upper bound on maximum rank

Blekherman-Teitler 2015: $r_{\max}(\mathbf{n}) \leq 2\mathrm{grank}(\mathbf{n})$

by showing that any tensor is a sum of two generic tensors

Know results

p	1	2	3	4	5	6	7	8	9
$r_{\max}(3,3,p)$	3	4	5	6	$\{6,7\}$	7	8	8	9

p	1	2	3	4	5	6	7	8	9
$grank(3,3,p)$	3	3	5	5	5	6	7	8	9

Rank of $2 \times p \times q$ tensors for $2 \leq p \leq q$

3-tensor $\mathcal{T} \in \mathbb{C}^{n \times p \times q}$ can be viewed as an album of

n photos $T_1, \ldots, T_n \in C^{p \times q}$,

change of basis in $\mathbb{C}^n, \mathbb{C}^p, \mathbb{C}^q$ corresponds

$A \times B \times C \in \mathbf{GL}(\mathbb{C}^n) \times \mathbf{GL}(\mathbb{C}^p) \times \mathbf{GL}(\mathbb{C}^q)$: $(I, B, C)(T_i) = BT_iC^\top$

action of $A = [a_{ij}]$ corresponds to $T_i' = \sum_{j=1}^n a_{ij} T_j$

Kronecker 1890 gave a canonical form of $B(T_1, T_2)C^\top$

JáJá 1979 determined the rank of $\mathcal{T}$ using Kronecker's form

$\mathrm{grank}(2, p, q) = \min(q, 2p)$ for $2 \leq p \leq q$

noncritical range: $p = 1 + (2-1)(p-1) \leq q$

$r_{\max}(2, p, q) = p + \lfloor \min(q, 2p)/2 \rfloor$ Atkinson-Stephens 1979

Kruskal's theorem applies for $r \leq \lfloor (p+q)/2 \rfloor$

Strassen's direct sum conjecture

Assume $\mathcal{S} \in \mathbb{C}^{\mathbf{m}}, \mathcal{T} \in \mathbb{C}^{\mathbf{n}}$, $\mathbf{m} = (m_1, \ldots, m_d), \mathbf{n} = (n_1, \ldots, n_d)$

View $\mathcal{S} \oplus \mathcal{T}$ as a tensor in $\mathbb{C}^{\mathbf{m}+\mathbf{n}}$

Clearly $\mathrm{rank}\ (\mathcal{S} \oplus \mathcal{T}) \le \mathrm{rank}\ \mathcal{S} + \mathrm{rank}\ \mathcal{T}$

For matrices ($d = 2$) equality holds

Strassen 1973 conjectured equality, Shitov disproved 2017 for $d = 3$

JáJá-Takche 1986 proved Strassen conjecture for $d = 3$ and

either $2 \in \{m_1, m_2, m_3, n_1, n_2, n_3\}$

or $2 \in \{m_i m_j - m_k, n_i n_j - m_k\}$ for some i, j, k s.t. $\{i, j, k\} = \{1, 2, 3\}$.

Buczyński-Postinghel-Rupniewski 2020 proved extra cases for $d = 3$

$\mathrm{rank}\ \oplus^k \mathcal{T} = k\mathrm{rank}\ \mathcal{T}$ if $\mathcal{T} \in \mathbb{C}^{(n_1,n_2,n_3)}$ satisfies J-T or B-P-R conditions

Kronecker tensor product

$\mathbf{U} := \otimes_K^{i\in[l]} \mathbf{V}_i$ viewing $\mathbf{U}$ as a vector space of dimension $\prod_{i=1}^{l} \dim \mathbf{V}_i$

Kronecker product of two matrices $A \otimes_K B = [a_{ij}] \otimes_K B = [a_{ij}B]$,

$A \in \mathbb{C}^{m_1 \times n_1}, B \in \mathbb{C}^{m_2 \times n_2}$, acts on $\mathbb{C}^{n_1} \otimes_K \mathbb{C}^{n_2} \cong \mathbb{C}^{n_1 n_2}$

$\otimes_K^{i\in[l]} \big(\otimes_{j=1}^{d} \mathbf{V}_{j,i} \big) := \otimes_{j=1}^{d} \big(\otimes_K^{i\in[l]} \mathbf{V}_{j,i} \big)$ is d-tensor space

Example $d = 3$ and $l = 2$:

Alice, Bob and Charlie $j \in \{1, 2, 3\}$ each have two particles $i \in \{1, 2\}$

$\mathbf{U} = (\mathbf{V}_{1,1} \otimes_K \mathbf{V}_{1,2}) \otimes (\mathbf{V}_{2,1} \otimes_K \mathbf{V}_{2,2}) \otimes (\mathbf{V}_{3,1} \otimes_K \mathbf{V}_{3,2})$ unfolding of

$\mathbf{V} = \otimes_{j=1}^{3} (\mathbf{V}_{j,1} \otimes \mathbf{V}_{j,2}) \cong \big(\otimes_{j=1}^{3} \mathbf{V}_{j,1} \big) \otimes \big(\otimes_{j=1}^{3} \mathbf{V}_{j,2} \big)$

$\mathcal{T} \in \mathbf{V} \Rightarrow \mathcal{T}' \in \mathbf{U}$ and $\mathrm{rank}\ \mathcal{T}' \le \mathrm{rank}\ \mathcal{T}$, $\mathrm{rank}\ (\mathcal{S} \otimes \mathcal{T}) \le (\mathrm{rank}\ \mathcal{S})(\mathrm{rank}\ \mathcal{T})$

Theorem $\mathrm{rank}\ \mathcal{W}_3 \otimes_K \mathcal{W}_3 = 7 < \mathrm{rank}\ \mathcal{W}_3 \otimes \mathcal{W}_3 = 8 < (\mathrm{rank}\ \mathcal{W}_3)^2 = 9$

1st eq. Chen-Chitambar-Duan-Winter 2010, 2nd eq. Chen-Friedland 2018

Symmetric tensors (Bosons in physics)

Symmetric d-mode tensor $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n$: $n_1 = \cdots = n_d = n$,

entries $s_{i_1,\ldots,i_d}$ are symmetric in all indexes;

$s_{i_{\sigma(1)},\ldots,i_{\sigma(d)}} = s_{i_1,\ldots,i_d}$ for a permutation $\sigma : [d] \to [d]$

$\mathrm{rank}\ T(\mathcal{S}) = \mathrm{rank}\ T_k(\mathcal{S})$ (unfolding in k-mode) for all $k \in [d]$

rank one symmetric tensor $\mathbf{x}^{\otimes^d} (= \otimes^d \mathbf{x}) := \mathbf{x} \otimes \cdots \otimes \mathbf{x} \neq 0$

symmetric rank (Waring rank): $\mathrm{srank}\,\mathcal{S} := \min\{r, \quad \mathcal{S} = \sum_{k=1}^r \varepsilon_k \mathbf{x}_k^{\otimes^d}\}$

$\varepsilon_k = 1$ *If* $\mathbb{F} = \mathbb{C}$, $\varepsilon_k = \pm 1$ *If* $\mathbb{F} = \mathbb{R}$ and d even

$\mathrm{sgen}(n^{\times d})$ -symmetric generic rank over

Alexander-Hirshowitz 1995: $\mathrm{sgen}(n^{\times d}) = \lceil \binom{n+d-1}{d} / d \rceil = \mathrm{sgrank}_0(n^{\times d})$

Exceptions: $n = 3, d = 4,\ n = 4, d = 4,\ n = 5, d = 3,\ n = 5, d = 4$

$\mathrm{sgrank}(n^{\times d}) = \mathrm{sgrank}_0(n^{\times d}) + 1$

Computation of symmetric tensor rank

For $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$, $\mathrm{srank}\, \mathcal{S} > r$ iff the system $\sum_{i=1}^r \mathbf{x}_i^{\otimes d} - \mathcal{S} = 0$ not solvable

Number of variables $M(r, n) = rn$, number of equations $s = \binom{n+d-1}{n-1}$

Number of linear coefficients for the system $\sum_{l=1}^s a_l(\mathbf{z}) f_l(\mathbf{z}) = 1$:

$\leq O((d+1)^{n-1} d^{(nr)(nr-1)})$

Number of linear equations for monomial coefficients $\leq d^{(nr)(nr-1)}$

Gauss elimination needs $O(\binom{n+d-1}{n-1} d^{3(nr)(nr-1)})$ flops

Corollary: for a fixed n, r finding if $\mathrm{rank}\, \mathcal{S} \leq r$ is polynomial in d

Maximum rank and identifiability of symmetric tensors

$r_{\max}(d, n)$-the maximum rank of (d, n) symmetric tensors

Buczyński-Han-Mella-Teitler 2018: $r_{\max}(d, n) \le 2r_{\text{gen}}(d, n) - 1$

$r_{\max}(d, 2) = d$, $r_{\text{gen}}(d, 2) = \left\lceil \frac{d+1}{2} \right\rceil$, $r_{\max}(3, 3) = 5$, $r_{\text{gen}}(3, 3) = 4$,

$r_{\max}(4, 3) = 7$, $r_{\text{gen}}(4, 3) = 6$, $r_{\max}(5, 3) = 10$, $r_{\text{gen}}(5, 3) = 7$,

$r_{\max}(3, 4) \ge 7$

Chiantini-Ottaviani-Vannieuwenhoven 2017: for $r < r_{\text{gen}}(d, n)$

a symmetric tensor of symmetric rank r in general position has unique symmetric rank decomposition, except in the cases:

$d = 6, n = 3$ and $r = 9$, $d = 4, n = 4$ and $r = 8$, $d = 3, n = 6$ and $r = 9$

Homogeneous polynomials & symmetric tensors

$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, $[n] = \{1, \ldots, n\}$

$f(\mathbf{x}) = \sum_{j_k+1 \in [d+1], k \in [n], j_1+\cdots+j_n = d} \frac{d!}{j_1! \cdots j_n!} f_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n} \in \mathrm{P}(d, n, \mathbb{F})$

$f(\mathbf{x}) = \sum_{\mathbf{j} \in J(d,n)} c(\mathbf{j}) f_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}, \ \mathbf{x}^{\mathbf{j}} = x_1^{j_1} \cdots x_n^{j_n}, \ c(\mathbf{j}) = \frac{d!}{j_1! \cdots j_n!}$

$J(d, n) = \{\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n, \ j_1 + \cdots + j_n = d\}, |J(d, n)| = \binom{n+d-1}{n-1}$

$\dim \mathrm{P}(d, n, \mathbb{F}) = \binom{n+d-1}{n-1}$, $\dim \mathrm{P}(d, 2, \mathbb{F}) = d + 1$

$\dim \mathrm{P}(d, n, \mathbb{F}) = O(d^{n-1})$ for a fixed value of n

$\mathrm{P}(d, n, \mathbb{F}) \sim \mathbb{F}^{J(d,n)} = \{\mathbf{f} = (f_{\mathbf{j}}), \mathbf{j} \in J(d, n)\}$

$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{\mathbf{j} \in J(d,n)} c(\mathbf{j}) f_{\mathbf{j}} \bar{g}_{\mathbf{j}}, \quad \|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$ HIlbert-Schmidt norm

Connection to symmetric tensors: $\mathcal{S} = [\mathcal{S}_{i_1,\ldots,i_d}] \in \mathrm{S}^d \mathbb{F}^n$

$f(\mathbf{x}) = \mathcal{S} \times \mathbf{x}^{\otimes d} = \sum_{i_j, i \in [d]} \mathcal{S}_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d}, \qquad f_{j_1,\ldots,j_n} = \mathcal{S}_{i_1,\ldots,i_d}$

j_l-the number of times $l \in [n]$ appears in $\{i_1, \ldots, i_d\}$

Skew symmetric tensors (Fermions in physics)

$\bigwedge^d \mathbb{F}^n \subset \mathbb{F}^{n^{\times d}}$ subspace of skew symmetric tensors, dim $\bigwedge^d \mathbb{F}^n = \binom{n}{d}$

Exterior algebra

$\mathcal{T} \in \bigwedge^d \mathbb{F}^n$ spanned by $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d = \sum_{\sigma\in\Pi(d)} \mathrm{sign}(\sigma)\mathbf{x}_{\sigma(1)} \wedge \cdots \wedge \mathbf{x}_{\sigma(d)}$

Note: $0 \neq \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d = \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_d, \langle \mathbf{y}_i, \mathbf{y}_j\rangle = c^2\delta_{ij}, c > 0$

$\mathrm{span}(\mathbf{x}_1, \ldots, \mathbf{x}_d) = \mathrm{span}(\mathbf{y}_1, \ldots, \mathbf{y}_d)$

dimension of Grassmanian $\mathrm{Gr}(d, n, \mathbb{F}) = (n - d)d$

arank $\mathcal{T}$ - min # terms in decomp. of $\mathcal{T}$ as sum of wedge products

minimal number of d-dimensional subspaces in $\mathbb{F}^n$ representing $\mathcal{T}$

$d!\,\mathrm{arank}\,\mathcal{T} \geq \mathrm{rank}\,\mathcal{T}$

$\mathrm{agrank}(n^{\times d}) \geq \lceil \frac{\binom{n}{d}}{(n-d)d+1} \rceil$ (affine parameter count)

How good this bound is for $3 \leq d \leq n/2$ (Duality)?

Spectral norm of tensors

For $\mathcal{T} \in \otimes_{j=1}^{d} \mathbb{F}^{n_j}$ the spectral norm is

$$\|\mathcal{T}\|_{\mathrm{spec}\,,\mathbb{F}} = \max\{|\mathcal{T} \times (\otimes_{j=1}^{d} \mathbf{x}_j)|,\ \mathbf{x}_j \in \mathbb{F}^{n_j}, \|\mathbf{x}_j\| = 1, j \in [d]\}$$

Banach's theorem-1938: for symmetric $\mathcal{S} \in \mathrm{S}^d \mathbb{F}^n$

$$\|\mathcal{S}\|_{\mathrm{spec}\,,\mathbb{F}} = \max\{|\mathcal{S} \times (\mathbf{x}^{\otimes^d})|,\ \mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| = 1\}$$

Redis.: Hübener-Kleinmann-Wei-González-Guillén-Gühne 2009 ($\mathbb{C}$)

Chen-He- Li-Zhang, Zhang-Ling-Qi 2012, Friedland 2013 ($\mathbb{R}$)

Analog of Banach's theorem for Fermions

For $\mathcal{T} \in \bigwedge^d \mathbb{F}^n$

$$\|\mathcal{T}\|_{spec,\mathbb{F}} = \max\{|\mathcal{T} \times (\otimes_{j=1}^d \mathbf{x}_j)|, \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}\} =$$

$$\max\{\frac{1}{d!}|\mathcal{T} \times \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d|, \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}\}$$

The value of spectral norm depends on $\mathbb{F}$

Spectral norm of real tensor over $\mathbb{C}$ maybe higher than its norm over $\mathbb{R}$:

$$\mathcal{T} = \frac{1}{2}\left(\mathbf{e}_1^{\otimes 2} \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_1^{\otimes 2} - \mathbf{e}_2^{\otimes 3}\right) \in S^3\mathbb{R}^2$$

Use Banach's theorem to show:

$$\|\mathcal{T}\|_{spec,\mathbb{R}} = \frac{1}{2}, \quad \|\mathcal{T}\|_{spec,\mathbb{C}} = \frac{1}{\sqrt{2}}$$

NP-hardness of computation of spectral norm

NP-hard to compute spectral norm for $d \geq 3$: Hillar-Lim 2013

Friedland-Lim 2018:

NP-hard to compute spectral norm of symmetric tensors for $d = 4$:

To compute clique number $\kappa(G)$ of undirected graph is NP-complete

undirected graph: $f(\mathbf{x}) = 2\sum_{1\le i<j\le n} a_{ij}x_i^2x_j^2$, $a_{ij} \in \{0, 1\}$

Motzkin-Strauss 1965: spectral norm of symmetric tensor $\mathcal{S} \in \mathrm{S}^4\mathbb{R}^n$

corresponding to f is $1 - \frac{1}{\kappa(G)}$

$\|\mathcal{S}\|_{spec,\mathbb{R}} = \|\mathcal{S}\|_{spec,\mathbb{C}}$

Computation of spectral norm of symmetric tensors I

With $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ associate $f(\mathbf{x}) = \mathcal{S} \times \mathbf{x}^{\otimes d} \in \mathrm{P}(d, n, \mathbb{C})$

Critial points of $\Re f(\mathbf{x})$ on $\|\mathbf{x}\| = 1$ satisfy

$\mathbf{F}(\mathbf{y}) = \lambda\bar{\mathbf{y}},\ \|\mathbf{y}\| = 1, \quad \mathbf{F}(\mathbf{x}) = \frac{1}{d}\nabla f(\mathbf{x}) = \mathcal{S} \times \mathbf{x}^{\otimes(d-1)}, \quad \bar{\mathbf{F}}(\mathbf{x}) = \overline{\mathbf{F}(\bar{\mathbf{x}})}$

$\mathbf{H}(\mathbf{x}) = \bar{\mathbf{F}}(\mathbf{F}(\mathbf{x}))$ critical points: $\mathbf{H}(\mathbf{y}) = \lambda^{d-1}\mathbf{y}$

which equivalent to nonzero fixed point $\mathbf{H}(\mathbf{z}) = \mathbf{z}$

For most $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ (whose hyperdeterminant nonzero)

$\big(\mathbf{F}(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}\big) \Rightarrow \big(\mathbf{H}(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{x} = 0\big)$

$\mathbf{H}(\mathbf{z}) - \mathbf{z} = \mathbf{0}$ has $(d-1)^{2n}$ finite solutions counting with multiplicities

$\mathbf{z} = \mathbf{0}$ is always isolated zero of multiplicity one

Computation of spectral norm of symmetric tensors II

Thee exists an extensive literature and software

on solving polynomial equations with isolated roots

Friedland-Wang 2020: Bit complexity of computation

within precision $\delta > 0$ is $O(d^{8n})$

Corollary: For a fixed n the complexity of finding spectral norm

is polynomial in d

The most entagled 3-qubit and 4-qubit

Chen-Xu-Zhu 2010: $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

$\||W\rangle\|_{spec} = \frac{2}{3}$ ($< \frac{1}{\sqrt{2}}$-spectral norm of most entagled 2-qubit)

Higuchi-Sudbery conjecture 2000 most entangled 4-qubit

$\mathcal{M}_4 = \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle + \omega(|1010\rangle + |0101\rangle) + \omega^2(|1001\rangle + |0110\rangle))$

$\omega = e^{2\pi\mathbf{i}/3}$

$\|\mathcal{M}_4\|_{spec} = \frac{\sqrt{2}}{3}$

Proved by: Derksen-Friedland-Lim-Wang 2017

Most equidimensional states are maximally entangled

$1 = \|\mathcal{T}\|^2 = \sum_{i_1 \in [n_1], \ldots, i_d \in [n_d]} |t_{i_1,\ldots,i_d}|^2 \Rightarrow$

$\|\mathcal{T}\|_{spec} \geq \|\mathcal{T}\|_{\ell_\infty} = \max |t_{i_1,\ldots,i_d}| \geq \frac{1}{\sqrt{N(\mathbf{n})}}, \quad N(\mathbf{n}) = n_1 \cdots n_d$

Equidimensional tensors: $n^{\times d} = (n, \ldots, n) \in \mathbb{N}^d$- d-qunits

$d \geq -2\log_2 \|\mathcal{T}\|_{\ell_\infty}, \quad \mathcal{T} \in \mathbb{C}^{n^{\times d}}, \|\mathcal{T}\| = 1$

Haar measure on $\mathrm{S}(n^{\times d}) = \{\mathcal{T} \in \mathbb{C}^{n^{\times d}}, \|\mathcal{T}\| = 1\}$

Gross-Flammia-Eisert 2009 for d-qubits

$\mathbf{P}\Big(-2\log_2 \|\mathcal{T}\|_\infty) \geq d - 2\log_2(d) - 3\Big) \geq 1 - e^{-d^2}, \qquad \text{for } d \geq 11$

Similar results for d-qunits Derksen-Makam 2020

Most symmetric states are maximally entangled

Bosons: $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n, \|\mathcal{S}\| = 1$

Dicke basis $\mathcal{D}_{1\le i_1\le\cdots\le i_d\le n} = \frac{1}{\sqrt{c(i_1,\ldots,i_d,n)}} \sum_{\sigma:[d]\to[d]} \otimes_{j=1}^d \mathbf{e}_{i_{\sigma(j)}}$

$\dim \mathrm{S}^d\mathbb{C}^n = d_{n,d} =: \binom{n+d-1}{d}$, $\|\mathcal{S}\|_\infty = \max\{\sqrt{c(i_1,\ldots,i_d,n)}|s_{i_1,\ldots,i_d}|\}$

Friedland-Kemp 2018:

$-2\log_2 \|\mathcal{S}\|_\infty \le \log_2 d_{n,d}$

$\mathbf{P}\Big(-2\log_2 \|\mathcal{S}\|_\infty \ge \log_2 d_{n,d} - \log_2\log_2 d_{n,d} - 3\log_2 n - 3\Big)$

$\ge 1 - (d_{n,d})^{-5n^3}$

Qubit bosons $n = 2$

$\mathbf{P}\Big(-2\log_2 \|\mathcal{S}\|_\infty \ge \log_2(d+1) - \log_2\log_2(d+1) - 3\log_2 2 - 3\Big)$

$\ge 1 - (d+1)^{-40}$

Nuclear norm of tensors

The nuclear norm of a tensor is the dual of the spectral norm:

$\|\mathcal{T}\|_{nuc,\mathbb{F}} = \max\{\Re(\mathcal{T}\times\mathcal{X}), \quad \|\mathcal{X}\|_{spec,\mathbb{F}} \le 1\}$

For matrices $d = 2$ $\|T\|_{nuc,\mathbb{F}} = \sum \sigma_i(T)$

$\mathrm{B}_{nuc,\mathbb{F}} := \{\mathcal{T} \in \otimes^d \mathbb{F}^{n_i}, \ \|\mathcal{T}\|_{nuc,\mathbb{F}} \le 1\}$ - the unit ball of the nuclear norm

$\otimes_1^d \mathbf{x}_i, \prod_{i=1}^d \|\mathbf{x}_i\| = 1, \mathbf{x}_i \in \mathbb{F}^{n_i}, i \in [d]$ are extreme points of $\mathrm{B}_{nuc,\mathbb{F}}$

Minimal characterization of the nuclear norm, minimal energy:

$\|\mathcal{T}\|_{nuc,\mathbb{F}} = \min\{\sum_{i=1}^k \prod_{j=1}^d \|\mathbf{x}_{j,i}\|, \quad \sum_{i=1}^k \otimes_{j=1}^d \mathbf{x}_{j,i} = \mathcal{T}\}$

Nuclear norm of Bosons and Fermions

Friedland-Lim analog of Banach's theorem for symmetric tensors

$\|\mathcal{S}\|_{nuc,\mathbb{F}} = \min\{\sum_{i=1}^{k} \|\mathbf{x}_i\|^d, \mathcal{S} = \sum \varepsilon_i \mathbf{x}_i^{\otimes^d}\}$

$\varepsilon_i = \pm 1$ if $\mathbb{F} = \mathbb{R}$ and d even, otherwise $\varepsilon_i = 1$

$\|\mathcal{S}\|_{nuc,\mathbb{F}}$ is NP-Hard to compute for $d \geq 3$,

follows from duality Friedland-Lim 2016

Characterization of $\|\mathcal{T}\|_{spec}$ of $\mathcal{T} \in \bigwedge^d \mathbb{F}^n$ yields

$\|\mathcal{T}\|_{nuc,\mathbb{F}} = \frac{1}{\sqrt{d!}} \min\{\sum_{i=1}^{k} \prod_{j=1}^{d} \|\mathbf{x}_{j,i}\|,$

$\mathcal{T} = \frac{1}{d!} \sum_{i=1}^{k} \mathbf{x}_{1,i} \wedge \cdots \wedge \mathbf{x}_{d,i}|, \langle \mathbf{x}_{j,i}, \mathbf{x}_{l,i} \rangle = \delta_{jl}\}$

Nuclear rank

nrank $\mathcal{T}$ - minimal number of components in nuclear decomposition:

$\mathcal{T} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i}, \|\mathcal{T}\|_{nuc,\mathbb{F}} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \|\mathbf{x}_{j,i}\|$

for matrices $d = 2$ nuclear rank is the rank

nrank $\mathcal{T} \ge \mathrm{rank}\ \mathcal{T}$, nrank is lower semicontinuous:

$\lim_{k\to\infty} \mathcal{T}_k = \mathcal{T} \Rightarrow \liminf_{k\to\infty}$ nrank $\mathcal{T}_k \ge$ nrank $\mathcal{T}$

No border rank phenomenon

Examples

$|GWZ\rangle = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$ is nuclear decomposition(unfold and use SVD)

$\mathcal{W}_3 = \mathbf{e}_1 \otimes e_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1$

$\|\mathcal{W}_3\| = 2/\sqrt{3}$ (easy), $\|\mathcal{W}_3\|_1 = (3\sqrt{3})/2$, nrank $\mathcal{W}_3 = 3$

$$\mathcal{W}_3 = \frac{\sqrt{3}}{2}\Bigg(\Big(\sqrt{\frac{2}{3}}\mathbf{e}_1 + \frac{1}{\sqrt{3}}\mathbf{e}_2\Big)^{\otimes 3} + \Big(\sqrt{\frac{2}{3}}\zeta\mathbf{e}_1 + \frac{1}{\sqrt{3}}\bar{\zeta}^2\mathbf{e}_2\Big)^{\otimes 3} + \Big(\sqrt{\frac{2}{3}}\bar{\zeta}\mathbf{e}_1 + \frac{1}{\sqrt{3}}\zeta^2\mathbf{e}_2\Big)^{\otimes 3}\Bigg), \; \zeta = e^{2\pi i/9}$$

Geometric interpretation of nuclear rank

Extreme points of $\mathrm{B}_{nuc,\mathbb{F}}$: $\mathrm{S}(n_1, \mathbb{F}) \times \cdots \mathrm{S}(n_d, \mathbb{F})$

Hence $\mathrm{B}_{nuc,\mathbb{F}}$ an orbitop:

The convex hall of the orbit of $\otimes \mathbf{e}_1^{\otimes d}$

under the action of $\mathbf{U}(n_1, \mathbb{F}) \times \cdots \times \mathbf{U}(n_d, \mathbb{F})$

nrank $\mathcal{T}$, $\|\mathcal{T}\|_{nuc,\mathbb{F}} = 1$ minimal decomposition of $\mathcal{T}$ on face of B_{nuc}

ngrank $\mathcal{T}$-minimal decomposition of generic $\mathcal{T}$ on a maximal facet

Dimension of maximal face: $\leq \mu(\mathbf{n}, \mathbb{R}) = N(\mathbf{n}) - 1 - \sum_{j=1}^{d} n_j(n_j - 1)/2$

$\leq \mu(\mathbf{n}, \mathbb{C}) = 2N(\mathbf{n}) - 1 - \sum_{j=1}^{d}(n_j^2 - n_j + 1)$

Caratheodory: $\mathrm{nrank}_{\mathbb{F}}(\mathcal{T}) \leq \mu(\mathbf{n}, \mathbb{F})$

References 1

- W. Bruzda, S. Friedland, K. Zyckowski, Rank of a tensor and quantum entanglement, arXiv:1912.06854.
- S. Friedland and T. Kemp, Most Boson quantum states are almost maximally entangled, *Proceedings of Amer. Math. Soc.* 146, No.12, (2018), 5035–5049.
- S. Friedland and L.-H. Lim, The computational complexity of duality, *SIAM Journal on Optimization*, 26, No. 4 (2016), 2378–2393.
- S. Friedland and L.-H. Lim, Nuclear norm of higher-order tensors, *Mathematics of Computation*, 87 (2018), 1255–1281.
- H. Derksen, S. Friedland, L.-H. Lim, and L. Wang, Theoretical and computational aspects of entanglement, arXiv:1705.07160.

References 2

- S. Friedland and L. Wang, Spectral norm of a symmetric tensor and its computation, *Mathematics of Computation*, 89 (2020)., 2175–2215.
- R. Sanyal, F. Sottile and B. Sturmfels, Orbitopes, *Mathematika* 57 (2011), no. 2, 275–314.