Andreas Kretschmer

When are (radicals of) symmetric ideals monomial?

14 November 2022

arXiv:2112.04464

Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Why symmetric ideals?

- Recent study of **asymptotic** or **limit behavior** of symmetric ideals in commutative algebra (noetherianity up to symmetry, ...).
- S_n-action probably the simplest "natural" finite group action on the polynomial ring (induced from GL-action).
- Connections to classical representation theory and combinatorics.
- Test case for solving polynomial systems with symmetries which are possibly less rigid.

Geometric point of view

Kleiman's theorem (special case)

- G connected algebraic group, action on variety X over $K = \overline{K}$.
- Action of G on X **transitive**.
- $V_1, V_2 \subseteq X$ closed subvarieties.

Then there exists non-empty Zariski open $G^\circ \subseteq G$ s.t. for all $g \in G^\circ$, $V_1 \cap g.V_2$ is either empty or pure of dimension

$$\dim(V_1) + \dim(V_2) - \dim(X).$$

Geometric point of view

Kleiman's theorem (special case)

- G connected algebraic group, action on variety X over $K = \overline{K}$.
- Action of G on X **transitive**.
- $V_1, V_2 \subseteq X$ closed subvarieties.

Then there exists non-empty Zariski open $G^\circ \subseteq G$ s.t. for all $g \in G^\circ$, $V_1 \cap g.V_2$ is either empty or pure of dimension

$$\dim(V_1) + \dim(V_2) - \dim(X).$$

Note: This holds for **any** V_1, V_2!

Geometric point of view

Kleiman's theorem (special case)

- G connected algebraic group, action on variety X over $K = \overline{K}$.
- Action of G on X **transitive**.
- $V_1, V_2 \subseteq X$ closed subvarieties.

Then there exists non-empty Zariski open $G^\circ \subseteq G$ s.t. for all $g \in G^\circ$, $V_1 \cap g.V_2$ is either empty or pure of dimension

$$\dim(V_1) + \dim(V_2) - \dim(X).$$

Note: This holds for **any** V_1, V_2!
Extensions of Kleiman's theorem to coherent sheaves, algebraically non-closed fields and not necessarily transitive actions exist.

Geometric point of view

Kleiman's theorem (special case)

- G connected algebraic group, action on variety X over $K = \overline{K}$.
- Action of G on X **transitive**.
- $V_1, V_2 \subseteq X$ closed subvarieties.

Then there exists non-empty Zariski open $G^\circ \subseteq G$ s.t. for all $g \in G^\circ$, $V_1 \cap g.V_2$ is either empty or pure of dimension

$$\dim(V_1) + \dim(V_2) - \dim(X).$$

Note: This holds for **any** V_1, V_2!
Extensions of Kleiman's theorem to coherent sheaves, algebraically non-closed fields and not necessarily transitive actions exist.
Instead: assumptions on V_1, V_2 become necessary. Sometimes hard to check.

S_n is discrete subgroup of GL_n $\Rightarrow$ not connected, not transitive.

S_n is discrete subgroup of $GL_n \Rightarrow$ not connected, not transitive.

Question: Could there be some sort of replacement for Kleiman's theorem?

S_n is discrete subgroup of GL_n $\Rightarrow$ not connected, not transitive.

Question: Could there be some sort of replacement for Kleiman's theorem?

Meta-theorem of Kleiman type for finite groups? (proposal)

1. G finite subgroup of $\mathrm{Aut}(X)$, X variety over $K = \overline{K}$.
2. Properties of action, restrictions for K (e.g. characteristic 0) and X (e.g. projective, smooth).
3. $V \subseteq X$ closed subscheme ranging in a certain variety $\mathcal{M}$ (e.g. hypersurfaces) of subschemes of X.

Then, for a **general** element $[V] \in \mathcal{M}$, the intersection

$$\bigcap_{g \in G} g.V$$

is "as small as it possibly could be" (e.g. empty).

S_n is discrete subgroup of GL_n $\Rightarrow$ not connected, not transitive.

Question: Could there be some sort of replacement for Kleiman's theorem?

Meta-theorem of Kleiman type for finite groups? (proposal)

1. G finite subgroup of $\mathrm{Aut}(X)$, X variety over $K = \overline{K}$.
2. Properties of action, restrictions for K (e.g. characteristic 0) and X (e.g. projective, smooth).
3. $V \subseteq X$ closed subscheme ranging in a certain variety $\mathcal{M}$ (e.g. hypersurfaces) of subschemes of X.

Then, for a **general** element $[V] \in \mathcal{M}$, the intersection

$$\bigcap_{g \in G} g.V$$

is "as small as it possibly could be" (e.g. empty).

My proposal for (2): *Few "small" G-representations*

In presence of a proper universal family

$$\mathcal{I} \subseteq \mathcal{M} \times X \xrightarrow{\mathrm{pr}_1} \mathcal{M},$$

such a theorem holds **at the level of dimensions** by applying Chevalley's theorem on upper semicontinuity of fiber dimensions to the morphism

$$\bigcap_{g \in G} g.\mathcal{I} \longrightarrow \mathcal{M}.$$

In presence of a proper universal family

$$\mathcal{I} \subseteq \mathcal{M} \times X \xrightarrow{\mathrm{pr}_1} \mathcal{M},$$

such a theorem holds **at the level of dimensions** by applying Chevalley's theorem on upper semicontinuity of fiber dimensions to the morphism

$$\bigcap_{g \in G} g.\mathcal{I} \longrightarrow \mathcal{M}.$$

But: can we say something at the level of **subsets** of X?

The case of S_n, concretely

$K = \overline{K}$.

Definition

A finite set $\mathcal{A}$ of monomials in $K[x_1, \ldots, x_n]$ is called **support set**.

- $\mathcal{A}$ is homogeneous if all its elements have same degree.
- $\mathcal{A}$ is symmetric if every permutation of its elements lies in $\mathcal{A}$ as well.

The case of S_n, concretely

$K = \overline{K}$.

Definition

A finite set $\mathcal{A}$ of monomials in $K[x_1, \ldots, x_n]$ is called **support set**.

- $\mathcal{A}$ is homogeneous if all its elements have same degree.
- $\mathcal{A}$ is symmetric if every permutation of its elements lies in $\mathcal{A}$ as well.

For $\mathcal{A}$ homogeneous:

- $X = \mathbb{P}^{n-1}$.
- $G = S_n$ with natural action on $\mathbb{P}^{n-1}$ permuting coordinates.
- $\mathcal{M} = \mathbb{P}(K^{\mathcal{A}})$, hypersurfaces of $\mathbb{P}^{n-1}$ defined by homogeneous polynomials with support set contained in $\mathcal{A}$.

Theorem [K. '22]

Let $n \geq 5$ and $\mathcal{A}$ homogeneous, symmetric, $\mathrm{char}(K) = 0$. Let $k \geq 1$ be the minimal number of distinct variables dividing a monomial in $\mathcal{A}$. Then, for a general $f \in K^{\mathcal{A}}$,

$$\sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k) \subseteq K[x_1, \ldots, x_n].$$

Equivalently, $\mathcal{V}(S_n.f) \subseteq \mathbb{P}^{n-1}$ is the union of all $(k-2)$-dimensional coordinate subspaces.

Theorem [K. '22]

Let $n \geq 5$ and $\mathcal{A}$ homogeneous, symmetric, $\mathrm{char}(K) = 0$. Let $k \geq 1$ be the minimal number of distinct variables dividing a monomial in $\mathcal{A}$. Then, for a general $f \in K^{\mathcal{A}}$,

$$\sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k) \subseteq K[x_1, \dots, x_n].$$

Equivalently, $\mathcal{V}(S_n.f) \subseteq \mathbb{P}^{n-1}$ is the union of all $(k-2)$-dimensional coordinate subspaces.

Conjecture

The same is true for all homogeneous support sets $\mathcal{A}$.

Conjecture

The same is true for all homogeneous support sets $\mathcal{A}$.
Moreover, for an arbitrary support set $\mathcal{A}$, the general $f \in K^{\mathcal{A}}$ satisfies

$$\mathcal{V}(S_n.f) \subseteq \mathcal{V}(S_n.x_1x_2\cdots x_k) \cup \mathcal{V}(x_i^e - x_j^e : i,j = 1,\ldots,n).$$

for some $e \in \mathbb{Z}_{>0}$ depending only on $\mathcal{A}$.

Proof of main theorem

Theorem [K. '22]

$K = \overline{K}$, $\mathrm{char}(K) = 0$. $\mathcal{A}$ homogeneous, symmetric in $n \geq 5$ variables, $k \geq 1$ minimal number of distinct variables dividing a monomial in $\mathcal{A}$. Then, for general $f \in K^{\mathcal{A}}$,

$$\mathcal{V}(S_n.f) = \mathcal{V}(S_n.x_1 x_2 \cdots x_k).$$

Few small S_n-representations

Main input: For $n \geq 5$ and $\mathrm{char}(K) = 0$, every irreducible S_n-representation of dimension > 1 has dimension $\geq n - 1$.

Few small S_n-representations

Main input: For $n \geq 5$ and $\mathrm{char}(K) = 0$, every irreducible S_n-representation of dimension > 1 has dimension $\geq n - 1$.

How is this useful?

Idea: Since $\mathcal{A}$ is symmetric, $K^{\mathcal{A}}$ is an S_n-representation! We view elements of $K^{\mathcal{A}}$ as coefficient vectors $(c_m)_{m \in \mathcal{A}}$ and write elements of the dual representation $(K^{\mathcal{A}})^*$ as $(y_m)_{m \in \mathcal{A}}$.

Idea: Since $\mathcal{A}$ is symmetric, $K^{\mathcal{A}}$ is an S_n-representation! We view elements of $K^{\mathcal{A}}$ as coefficient vectors $(c_m)_{m\in\mathcal{A}}$ and write elements of the dual representation $(K^{\mathcal{A}})^*$ as $(y_m)_{m\in\mathcal{A}}$.

Consider rational map

$$\varphi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}((K^{\mathcal{A}})^*),\ [x_1 : \cdots : x_n] \mapsto [m(x_1, \ldots, x_n)],$$

where $[m(x_1, \ldots, x_n)]$ denotes the homogeneous coordinate vector of all $m \in \mathcal{A}$.

Idea: Since $\mathcal{A}$ is symmetric, $K^{\mathcal{A}}$ is an S_n-representation! We view elements of $K^{\mathcal{A}}$ as coefficient vectors $(c_m)_{m\in\mathcal{A}}$ and write elements of the dual representation $(K^{\mathcal{A}})^*$ as $(y_m)_{m\in\mathcal{A}}$.

Consider rational map

$$\varphi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}((K^{\mathcal{A}})^*), \; [x_1 : \dots : x_n] \mapsto [m(x_1, \dots, x_n)],$$

where $[m(x_1, \dots, x_n)]$ denotes the homogeneous coordinate vector of all $m \in \mathcal{A}$.

Indeterminacy locus of φ is $\mathcal{V}(S_n.x_1x_2\cdots x_k)$. Complement denoted $U \subseteq \mathbb{P}^{n-1}$.

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m], [y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m],[y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

- Fiber of pr_1 over polynomial g is the image under φ of the solution set $\mathcal{V}(S_n.g)$.

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m],[y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

- Fiber of pr_1 over polynomial g is the image under φ of the solution set $\mathcal{V}(S_n.g)$.
- Fiber of pr_2 over point $[y_m]$ is the set of polynomials g having this "solution."

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m],[y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

- Fiber of pr_1 over polynomial g is the image under φ of the solution set $\mathcal{V}(S_n.g)$.
- Fiber of pr_2 over point $[y_m]$ is the set of polynomials g having this "solution."
  $\Rightarrow$ **Fibers of pr_2 are subrepresentations of $K^{\mathcal{A}}$!**

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m],[y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

- Fiber of pr_1 over polynomial g is the image under φ of the solution set $\mathcal{V}(S_n.g)$.
- Fiber of pr_2 over point $[y_m]$ is the set of polynomials g having this "solution."
  $\Rightarrow$ **Fibers of pr_2 are subrepresentations of $K^{\mathcal{A}}$!**
- The image $\mathrm{pr}_1(\mathcal{Z}) \subseteq \mathbb{P}(K^{\mathcal{A}})$ is the set of all g for which

$$\mathcal{V}(S_n.g) \neq \mathcal{V}(S_n.x_1 x_2 \cdots x_k).$$

Consider constructible subset $\mathcal{Z} \subseteq \mathbb{P}(K^{\mathcal{A}}) \times \mathbb{P}((K^{\mathcal{A}})^*)$ given by

$$\mathcal{Z} = \{([c_m],[y_m]) : [y_m] \in \varphi(U), \sum_{m \in \mathcal{A}} c_m y_{\sigma.m} = 0 \text{ for all } \sigma \in S_n\}.$$

- Fiber of pr_1 over polynomial g is the image under φ of the solution set $\mathcal{V}(S_n.g)$.
- Fiber of pr_2 over point $[y_m]$ is the set of polynomials g having this "solution."
  $\Rightarrow$ **Fibers of pr_2 are subrepresentations of $K^{\mathcal{A}}$!**
- The image $\mathrm{pr}_1(\mathcal{Z}) \subseteq \mathbb{P}(K^{\mathcal{A}})$ is the set of all g for which

$$\mathcal{V}(S_n.g) \neq \mathcal{V}(S_n.x_1 x_2 \cdots x_k).$$

$\Rightarrow$ Enough to prove $\dim \mathcal{Z} < \dim \mathbb{P}(K^{\mathcal{A}})$.

To prove this, write $\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_3$, where $\varphi(U) = T_1 \sqcup T_2 \sqcup T_3$ and $\mathcal{Z}_i = \mathrm{pr}_2^{-1}(T_i)$.

To prove this, write $\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_3$, where $\varphi(U) = T_1 \sqcup T_2 \sqcup T_3$ and $\mathcal{Z}_i = \mathrm{pr}_2^{-1}(T_i)$.

Idea: Fiber of $\mathcal{Z} \xrightarrow{\mathrm{pr}_2} \varphi(U)$ over each point $[y_m]$ is subrepresentation of $\mathbb{P}(K^{\mathcal{A}})$.

- T_1: Permutations of point $[y_m] \in T_1$ generate trivial/sign repr.
- T_2: Permutations of point $[y_m] \in T_2$ generate repr. containing irred. of dim. > 1.
- T_3: Permutations of point $[y_m] \in T_3$ generate repr. containing irred. of dim. > 1 and trivial/sign repr.

To prove this, write $\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_3$, where $\varphi(U) = T_1 \sqcup T_2 \sqcup T_3$ and $\mathcal{Z}_i = \mathrm{pr}_2^{-1}(T_i)$.

Idea: Fiber of $\mathcal{Z} \xrightarrow{\mathrm{pr}_2} \varphi(U)$ over each point $[y_m]$ is subrepresentation of $\mathbb{P}(K^{\mathcal{A}})$.

- T_1: Permutations of point $[y_m] \in T_1$ generate trivial/sign repr.
- T_2: Permutations of point $[y_m] \in T_2$ generate repr. containing irred. of dim. > 1.
- T_3: Permutations of point $[y_m] \in T_3$ generate repr. containing irred. of dim. > 1 and trivial/sign repr.

Then:

- $\dim T_1 = 0$, fibers contained in hyperplanes of $\mathbb{P}(K^{\mathcal{A}})$.
- $\dim T_2 \leq n-2$ and all fibers have codim $\geq n-1$.
- $\dim T_3 \leq n-1$ and all fibers have codim $\geq n$.

To prove this, write $\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_3$, where $\varphi(U) = T_1 \sqcup T_2 \sqcup T_3$ and $\mathcal{Z}_i = \mathrm{pr}_2^{-1}(T_i)$.

Idea: Fiber of $\mathcal{Z} \xrightarrow{\mathrm{pr}_2} \varphi(U)$ over each point $[y_m]$ is subrepresentation of $\mathbb{P}(K^{\mathcal{A}})$.

- T_1: Permutations of point $[y_m] \in T_1$ generate trivial/sign repr.
- T_2: Permutations of point $[y_m] \in T_2$ generate repr. containing irred. of dim. > 1.
- T_3: Permutations of point $[y_m] \in T_3$ generate repr. containing irred. of dim. > 1 and trivial/sign repr.

Then:

- $\dim T_1 = 0$, fibers contained in hyperplanes of $\mathbb{P}(K^{\mathcal{A}})$.
- $\dim T_2 \leq n - 2$ and all fibers have codim $\geq n - 1$.
- $\dim T_3 \leq n - 1$ and all fibers have codim $\geq n$.

$\Rightarrow \dim \mathcal{Z}_i < \dim \mathbb{P}(K^{\mathcal{A}})$.

1. Assume there is $m_0 \in \mathcal{A}$, not a power of $x_1 x_2 \cdots x_n$, s.t.

$$0 \neq y_{m_0} = y_{\sigma.m_0} \text{ for all } \sigma \in A_n.$$

$\longrightarrow$ Only finitely many such points in $\varphi(U)$. The fiber of $\mathcal{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{P}((K^{\mathcal{A}})^*)$ over each is contained in a hyperplane in $\mathbb{P}(K^{\mathcal{A}})$.

2. Assume there is $m_0 \in \mathcal{A}$, not a power of $x_1 x_2 \cdots x_n$, s.t. $\sum_{m \in S_n.m_0} y_m \neq 0$ and not all y_m with m in A_n-orbit of m_0 agree.

Claim: Every fiber of $\mathcal{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{P}((K^{\mathcal{A}})^*)$ over such point $[y_m]$ is linear subspace of $\mathbb{P}(K^{\mathcal{A}})$ of codim $\geq n$.

Indeed, this fiber is the kernel of the matrix Y whose rows are all S_n-permutations of $(y_m)_{m \in \mathcal{A}}$.

The codim of this kernel in $K^{\mathcal{A}}$ is $\mathrm{rk}(Y)$.

Claim: Even the submatrix Y' of Y whose rows are all the S_n-permutations of $(y_m)_{m \in S_n.m_0}$ has $\mathrm{rk}(Y') \geq n$.

$\longrightarrow$ Row span of Y' is subrepresentation of permutation module M^{m_0} containing the trivial representation – sum of rows – and some irreducible representation of dim $\geq n-1$.

3. Otherwise, for all $m_0 \in \mathcal{A}$, not a power of $x_1 x_2 \cdots x_n$, we have $\sum_{m \in S_n.m_0} y_m = 0$ and not all y_m with m in the A_n-orbit of m_0 agree.

This translates to codim 1 condition

$$\sum_{m \in S_n.m_0} m(x_1, \ldots, x_n) = 0$$

on $\mathbb{P}^{n-1}$.

$\Rightarrow$ Dimensions add up to $< \dim \mathbb{P}(K^{\mathcal{A}})$ in each case. □

Summary

We proved:

$$\mathcal{A} \text{ homogeneous, symmetric, } \mathrm{char}(K) = 0,\ n \geq 5$$
$$\Rightarrow \text{General } f \in K^{\mathcal{A}} \text{ satisfies } \sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k).$$

Summary

We proved:

$\mathcal{A}$ homogeneous, symmetric, $\mathrm{char}(K) = 0$, $n \geq 5$
$\Rightarrow$ General $f \in K^{\mathcal{A}}$ satisfies $\sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k)$.

We conjectured:

Symmetry assumption on $\mathcal{A}$ should be unnecessary.
Inhomogeneous version should exist.

Summary

We proved:

$\mathcal{A}$ homogeneous, symmetric, $\mathrm{char}(K) = 0$, $n \geq 5$
$\Rightarrow$ General $f \in K^{\mathcal{A}}$ satisfies $\sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k)$.

We conjectured:

Symmetry assumption on $\mathcal{A}$ should be unnecessary.
Inhomogeneous version should exist.

We asked:

Is there a general theorem (Kleiman-type) behind this phenomenon?

Summary

We proved:

$\mathcal{A}$ homogeneous, symmetric, $\mathrm{char}(K) = 0$, $n \geq 5$
$\Rightarrow$ General $f \in K^{\mathcal{A}}$ satisfies $\sqrt{(S_n.f)} = (S_n.x_1 x_2 \cdots x_k)$.

We conjectured:

Symmetry assumption on $\mathcal{A}$ should be unnecessary.
Inhomogeneous version should exist.

We asked:

Is there a general theorem (Kleiman-type) behind this phenomenon?

Thank you!