# Tensors decomposition and extension

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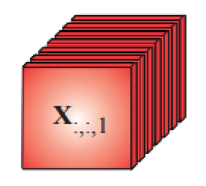


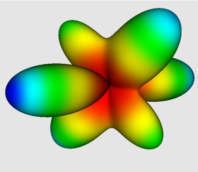

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- Used in **many applications** to recover this hidden information (signal processing, data analysis and machine learning, phylogenetic, chemistry, ...), to compress data and improve performance, to find better computational methods ...
- How to compute a **rank-revealing decomposition** of a tensor ?
- Can we compute robustly the rank ?

# Matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \ldots$

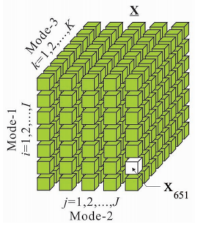
$M \in \mathbb{K}^{n_1 \times n_2} = \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2}$ is of **rank** $r$ iff there exist
$U \in \mathbb{K}^{n_1 \times n_1}$, $V \in \mathbb{K}^{n_2 \times n_2}$ invertible and $\Sigma_r$ diagonal invertible s.t.

$$M = U \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} V$$

- $\Sigma_r$ not unique
- $\Sigma_r = I_r$ for some $U, V$.
- $U, V$ unitary $\Rightarrow$ Singular Value Decomposition
- $U$, $V$ are **eigenvectors** of $M\,M^t$ (resp. $M^t M$)
- Best low rank approximation from truncated SVD

# Multilinear tensors

$$T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3} \quad = $$




$$\equiv \quad T(\mathbf{x}_1, \ldots, \mathbf{x}_d) \text{ multilinear in } \mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n_i})$$

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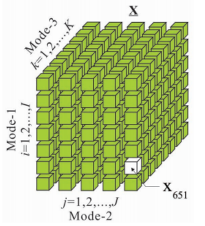


$\equiv \quad T(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ multilinear in $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n_i})$

**Multilinear tensor decomposition problem:**

Given a multilinear polynomial $T(\mathbf{x})$ of order $d$ in the variables $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ with coefficients $\in \mathbb{K}$ find a minimal decomposition of $T$ of the form

$$T(\mathbf{x}) = \sum_{i=1}^{r} u_{j,1}(\mathbf{x}_1) \cdots u_{j,d}(\mathbf{x}_d)$$

(Equivalently $T = \sum_{i=1}^{r} u_{j,1} \otimes \cdots \otimes u_{j,d}$).

The minimal $r$ in such a decomposition is called the **rank** of $T$.

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Given a homogeneous polynomial $T$ of degree $d$ in the variables $\overline{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

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$$T(\overline{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0}x_0 + \xi_{i,1}x_1 + \cdots + \xi_{i,n}x_n)^d$$

with $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

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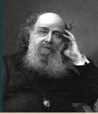
# Tensor decomposition

# Sylvester approach (1851)



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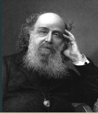

## Theorem

*The binary form* $T(x_0, x_1) = \sum_{i=0}^{d} t_i \binom{d}{i} x_0^{d-i} x_1^i$ *can be decomposed as a sum of* $r$ *distinct powers of linear forms*

$$T = \sum_{k=1}^{r} \omega_k (\alpha_{\mathbf{k}} x_0 + \beta_{\mathbf{k}} x_1)^d$$

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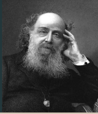

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$$T = \sum_{k=1}^{r} \omega_k (\alpha_{\mathbf{k}} x_0 + \beta_{\mathbf{k}} x_1)^d$$

*iff there exists a polynomial* $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\begin{bmatrix} t_0 & t_1 & \dots & t_r \\ t_1 & & & t_{r+1} \\ \vdots & & & \vdots \\ t_{d-r} & \dots & t_{d-1} & t_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

*and of the form* $p = c \prod_{k=1}^{r} (\beta_{\mathbf{k}} x_0 - \alpha_{\mathbf{k}} x_1)$ *with* $(\alpha_k : \beta_k)$ *distinct.*

If $\alpha_k \neq 0$, $\xi_k = \frac{\beta_k}{\alpha_k}$ root of $p(x) = \sum_{i=0}^{r} p_i x^i$.

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$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

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- Compute the roots $\xi_\mathbf{1} = \mathbf{e}^{\zeta_1}, \ldots, \xi_\mathbf{r} = \mathbf{e}^{\zeta_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.

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- Compute the roots $\boldsymbol{\xi_1 = e^{\zeta_1}}, \ldots, \boldsymbol{\xi_r = e^{\zeta_r}}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

to obtain the decomposition $\sigma_k = \sum_{i=1}^{r} \omega_i \xi_i^k$.

# Duality

$S^d(E)^* \equiv S^d(E^*) \equiv \mathbb{K}[x_1,\dots,x_n]_d = \mathbb{K}[\mathbf{x}]_d =: S_d.$

**Apolar product:** For $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha}\mathbf{x}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha}\mathbf{x}^\alpha \in S_d$,

$$\langle f, g\rangle_d = \sum_{|\alpha|=d} f_\alpha\, g_\alpha \binom{d}{\alpha}.$$

**Dualization:** For $F \in S_d$, $\mathbf{F}^* \in \mathbf{S}_\mathbf{d}^* = \mathrm{Hom}_{\mathbb{K}}(\mathbf{S}_\mathbf{d}, \mathbb{K}) : p \mapsto \langle F, p\rangle_d$

**Properties:**

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- ☞ $F = \sum_{i=1}^r w_i \, \xi_i^{\otimes d} \in S_d$ iff $F^* \in \langle \mathbf{e}_{\xi_1}^{[d]}, \ldots, \mathbf{e}_{\xi_r}^{[d]} \rangle \subset S_d^*$

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- Catalecticant $H_F^{k,d-k}$: $p \in S_{d-k} \mapsto p \star F^* \in S_k^*$
  where for $\psi \in S_d^*, p \in S_k$, $\quad p \star \psi : q \in S_{d-k} \mapsto \psi(pq)$

  (can also be def. by polarisation $v \in S^1(E) \mapsto v \otimes 1 + 1 \otimes v \in S^1(E) \otimes S^0(E) + S^0(E) \otimes S^1(E)$)

## Definition (Apolar ideal)

$F_k^{\perp} = \ker H_F^{d-k,k}, \quad (F^{\perp}) = \mathrm{Ann}(F^*) = (F_k^{\perp}, k \in 0{:}d)$

- If $F = \sum w_j \, \xi_j^{\otimes d}$ then, for $p \in S^k$ with $p(\xi_j) = 0$,

$$H_F^{d-k,k}(p) = [\mathbf{x}^{\beta} \star p \star F^*] = [\sum_j \omega_j \, \xi_j^{\beta} \, p(\xi_j)] = 0$$

and

$$F_k^{\perp} \supset I(\Xi)_k := \{p \in S^k \mid p(\xi_j) = 0, j = 1, \ldots, r\}.$$

- If $V(F_k^{\perp}) = \{\xi_1, \ldots, \xi_r\}$ then $F^* \in \langle \mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r} \rangle$ and $F = \sum_j \omega_j \xi^{\otimes d}$
- If $\mathrm{rank}\, H_F^{d-k,k} = \dim \langle \mathbf{e}_{\xi_1}^{[k]}, \ldots, \mathbf{e}_{\xi_r}^{[k]} \rangle = r$ for some $k \leq d$ then

$$\langle \mathbf{x}^{\beta} \star F^*; |\beta| = d-k \rangle = \langle \mathbf{e}_{\xi_1}^{[k]}, \ldots, \mathbf{e}_{\xi_r}^{[k]} \rangle = I(\Xi)_k^{\perp}.$$

## Definition (Interpolation degree $\iota(\xi_1, \ldots, \xi_r)$)

min. degree of interpolation polynomials = CM regularity of $I(\xi_1, \ldots, \xi_r)$ $-1$

## Theorem

$F = \sum_{i=1}^{r} \omega_i \, \xi_i^{\otimes d} \in S_d$ *with* $\iota(\xi_1, \ldots, \xi_r) < k := \lfloor \frac{d-1}{2} \rfloor$ **iff**
$\operatorname{rank} H_F^{d-k,k} = r$ *and there exists* $\mathbf{b} \subset S_k$, $\mathbf{b}' \subset S_{d-k-1}$ *of size* $r$,
$U, V \in \mathbb{K}^{r \times r}$ *invertible and* $\Sigma_i$ *diagonal s.t.*

$$H_F^{\mathbf{b}, x_i \mathbf{b}'} = U \, \Sigma_i \, V \qquad \text{for } i \in 1{:}n$$

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- $\mathrm{rank}\, F = r$
- $F$ is **identifiable** (unique decomp. up to perm. and scaling of $\xi_i$)
- $V^{-1}$ = **interpolation polynomials** at $\xi_1, \ldots, \xi_r$ (up to scaling)
- $U$ = **evaluations $\mathbf{e}_{\xi_i}^{[d-k]}$** for $i \in 1{:}r$ (up to scaling)
- $\xi_i = ((\Sigma_j)_{i,i})_{j \in 1:n}$, $i \in 1{:}r$
- $F_k^{\perp} = I(\xi_1, \ldots, \xi_r)_k$ (generators of the vanishing ideal)

# Symmetric tensor decomposition

$$\begin{aligned}
F &= (x_0 + 3\,x_1 - x_2)^4 + (x_0 + x_1 + x_2)^4 - 3\,(x_0 + 2\,x_1 + 2\,x_2)^4 \\
&= -{x_0}^4 - 8\,{x_0}^3 x_1 - 24\,{x_0}^3 x_2 - 60\,{x_0}^2 {x_2}^2 - 168\,{x_0}^2 x_1 x_2 - 12\,{x_0}^2 {x_1}^2 \\
&\quad -96\,x_0 {x_2}^3 - 240\,x_0 x_1 x_2^2 - 384\,x_0 x_1^2 x_2 + 16\,x_0 {x_1}^3 - 46\,{x_2}^4 - 200\,x_1 x_2^3 \\
&\quad -228\,{x_1}^2 {x_2}^2 - 296\,x_1^3 x_2 + 34\,{x_1}^4 \\
\langle F, p\rangle_4 &= \langle F^* | p\rangle \text{ where } F^* = \mathbf{e}_{(1,3,-1)} + \mathbf{e}_{(1,1,1)} - 3\,\mathbf{e}_{(1,2,2)} \quad (\textit{by apolarity})
\end{aligned}$$

$$H_F^{2,2} := \begin{bmatrix}
-1 & -2 & -6 & -2 & -14 & -10 \\
-2 & -2 & -14 & 4 & -32 & -20 \\
-6 & -14 & -10 & -32 & -20 & -24 \\
-2 & 4 & -32 & 34 & -74 & -38 \\
-14 & -32 & -20 & -74 & -38 & -50 \\
-10 & -20 & -24 & -38 & -50 & -46
\end{bmatrix}$$

For $\mathbf{b}' = \{x_0, x_1, x_2\}, \mathbf{b} = x_0 \mathbf{b}'$

$$H_F^{\mathbf{b}, x_0 \mathbf{b}'} = \begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix}$$

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- The matrix of multiplication by $x_2 x_0^{-1}$ in $x_0\,\mathbf{b}' = \{x_0^2, x_0x_1, x_0x_2\}$ is

$$M_2 = (H_F^{\mathbf{b},x_0\mathbf{b}'})^{-1} H_F^{\mathbf{b},x_2\mathbf{b}'} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

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- Its eigenvalues are $[\mathbf{-1}, \mathbf{1}, \mathbf{2}]$ and the eigenvectors:

$$V := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

  that is the polynomials

$$V(\mathbf{x}) = \left[\ \frac{1}{2}\, x_1 - \frac{1}{2}\, x_2 \quad -2\, x_0 + \frac{3}{4}\, x_1 + \frac{1}{4}\, x_2 \quad -x_0 + \frac{1}{2}\, x_1 + \frac{1}{2}\, x_2\ \right].$$

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- We deduce the weights and the frequencies:

$$H_F^{\mathbf{b},V} = \begin{bmatrix} 1 & 1 & -3 \\ 1 \times \mathbf{3} & 1 \times \mathbf{1} & -3 \times \mathbf{2} \\ 1 \times \mathbf{-1} & 1 \times \mathbf{1} & -3 \times \mathbf{2} \end{bmatrix}$$

Weights: $1, 1, -3$;

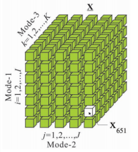
Frequencies: $(-1, 3), (1, 1), (2, 2)$.

**Decomposition:**

$$\begin{aligned} F^* &= \mathbf{e}_{(3,-1)} + \mathbf{e}_{(1,1)} - 3\,\mathbf{e}_{(2,2)} \\ F &= (\mathbf{x_0} + \mathbf{3}\,\mathbf{x_1} - \mathbf{x_2})^4 + (\mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2})^4 - 3\,(\mathbf{x_0} + \mathbf{2}\,\mathbf{x_1} + \mathbf{2}\,\mathbf{x_2})^4 \end{aligned}$$

# Decomposition of multilinear tensors

$T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3} \quad \equiv \quad [T_{[i]}]_{i=1}^{n_3}$ pencil of $n_3$ matrices of size $n_1 \times n_2$.

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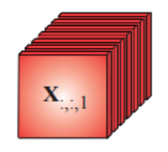
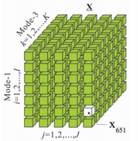


# Decomposition of multilinear tensors

$T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3} \quad \equiv \quad [T_{[i]}]_{i=1}^{n_3}$ pencil of $n_3$ matrices of size $n_1 \times n_2$.

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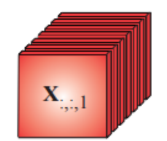


For $T \in \mathbb{K}^{\mathbf{r}} \otimes \mathbb{K}^{\mathbf{r}} \otimes \mathbb{K}^{n_3}$,

$$T = \sum_{j=1}^{\mathbf{r}} U_j \otimes V_j \otimes W_j \text{ with } U, V \in \mathbb{K}^{\mathbf{r}\times\mathbf{r}}, W \in \mathbb{K}^{n_3\times\mathbf{r}}$$

$$\text{iff} \quad \mathrm{T}_{[i]} = U \operatorname{diag}(W_{i,1}, \ldots, W_{i,\mathbf{r}}) V^t \quad i \in 1{:}n_3$$

If $T_{[1]}$ inv., $\quad U =$ matrix of **common eigenvectors** of $M_i = T_{[i]} T_{[1]}^{-1}$

$V^{-t} =$ matrix of **common eigenvectors** of $M_i' = T_{[1]}^{-1} T_{[i]}$ .

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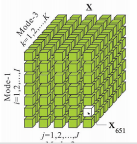


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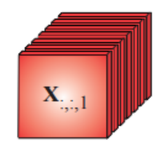


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**Decomposition** (when $T_0 = \sum_i l_i T_{[i]}$ invertible):

- Compute the common eigenvectors $U$ of $M_i = T_{[i]} T_0^{-1}$ for $T_0 = \sum_i l_i T_{[i]}$;
- Deduce the common eigenvectors $V^{-t}\Sigma_0 = T_0^{-1} U$ of $M_i' = T_0^{-1} T_{[i]}$ and $\operatorname{diag}(W_{i,1}, \ldots, W_{i,r}) = U T_{[i]} V^{-t}$;

# Tensor extension

# Degree extension of tensors

## Definition

For $F \in S_d$ and $d' > d$, $\tilde{F} \in S_{d'}$ is a **degree-$d'$ extension** of $F$ if there exists $x_0 \in S_1$, $x_0 \neq 0$ s.t.

$$\tilde{F} = x_0^{d'-d} F + R$$

with $\deg_{x_0} R < d' - d$.

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- ☞ Set "$x_0 = 1$"

  $f = F^*_{|x_0=1} \in (S_d^*)_{|x_0=1} = \mathbb{K}[\mathbf{y}]_{\leq d}$ (dual monomials)

  $\tilde{f} = \tilde{F}^*_{|x_0=1} \in (S_{d'}^*)_{|x_0=1} = \mathbb{K}[\mathbf{y}]_{\leq d'}$

$$\tilde{f}(\mathbf{y}) = f(\mathbf{y}) + (\mathbf{y})^{d+1}$$

# Flat extension of Hankel matrices

For (monomial) sets $\mathbf{b} \subset \mathbf{c}$, $\mathbf{b}' \subset \mathbf{c}'$, $\overline{\mathbf{b}} = \mathbf{c} \setminus \mathbf{b}$, $\overline{\mathbf{b}}' = \mathbf{c}' \setminus \mathbf{b}'$.

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**Definition**

For $F \in S_d$, $d' > d$, $\tilde{F} \in S^{d'}$ is a **flat extension** of $F$ of rank $r$ if $\exists$ $x_0 \in S_1$,

- $\tilde{F}$ is a $x_0$ degree-$d'$ extension of $F$
- $\mathbf{b} \subset (x_0) \cap S_k$, $\mathbf{b}' \subset (x_0) \cap S_{d-k'}$ of size r s.t.

$$\operatorname{rank} H_{\tilde{F}}^{\mathbf{b},\mathbf{b}'} = \operatorname{rank} H_{\tilde{F}}^{k,d'-k} = r$$

**Definition:** For $\mathbf{b} \subset x_0 \cdot S_{d-1}$ and $\mathbf{b}^+ := x_0^{-1}\mathbf{x} \cdot \mathbf{b}$, we say that $\mathbf{b}$ is $x_0$-**connected** if $x_0^d \in \mathbf{b}$ and $x_i m \in \mathbf{b} \Rightarrow x_0 m \in \mathbf{b}$.

**Theorem**

*Let $F \in S^*_{d+d'}$, $\mathbf{b} \subset x_0 \cdot S_{d-1}, \mathbf{b}' \subset x_0 \cdot S_{d'-1}$ $x_0$-connected and $\mathbf{c} \supset \mathbf{b}^+, \mathbf{c}' \supset \mathbf{b}'^+$. Assume $H_F^{\mathbf{b},\mathbf{b}'}$ invertible with $|\mathbf{b}| = |\mathbf{b}'| = r$. The following points are equivalent:*

- *$H_F^{\mathbf{c},\mathbf{c}'}$ is a* **flat extension** *of $H_F^{\mathbf{b},\mathbf{b}'}$.*
- *The operators $M_j := H_F^{\mathbf{b},x_j x_0^{-1}\mathbf{b}'}(H_F^{\mathbf{b},\mathbf{b}'})^{-1}$* **commute**.

*In this case, $\tilde{F}$ is called an* **extensor** *of $F$.*

[Brachat-Comon-M-Tsigaridas'10; Brachat-Bernardi-Comon-M'11; Laurent-M'09]

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### Theorem

*$F \in S_d$ is of rank $r$ iff there exists a* **flat extension** *$\tilde{F} \in S_{d'}$ of $F$ of rank $r$ with $d' = m + m'$, $m = \max\{r-1, \lceil \frac{d}{2} \rceil\}$, $m' = \max\{r, \lfloor \frac{d}{2} \rfloor\}\}$.*

[Bernardi-Brachat-M'11; Buczynska-Buczynski'11]

If $\tilde{F}$ is a **flat extension** of $F$ of rank $r$, then

- ☞ $I := (\ker H_{\tilde{F}}^{k,d'-k})$ zero-dimensional of degree $r$.
- ☞ $I \subset (F^{\perp})$ is apolar to $F$.
- ☞ If $\mathcal{V}_{\overline{\mathbb{K}}}(I) \equiv \{\xi_1, \dots, \xi_r\}$ and $I$ is radical, then $F = \sum_{i=1}^{r} \xi_i^{\otimes d}$.
- ☞ If $\mathcal{V}_{\overline{\mathbb{K}}}(I) \equiv \{\xi_1, \dots, \xi_{r'}\}$ and the local inverse system of $I$ at $\xi_i$ is "generated" by $w_i \in S_{d_i}$ then $F = \sum_{i=1}^{r'} \xi_i^{\otimes d-d_i} w_i$ (General Additive Decomposition [Iarrobino-Kanev'99]).
- ☞ $\exists\ \mathbf{b} \subset S_m$, $x_0 \in S_1$, $\mathbf{b}' \subset x_0 \cdot S_{m'-1}$, $|\mathbf{b}| = |\mathbf{b}'| = r$ s.t. $\mathrm{rank}\, H_{\tilde{F}}^{\mathbf{b},\mathbf{b}'} = r$
- ☞ $\exists\ U, V \in \overline{\mathbb{K}}^{r\times r}$ invertible, $\Sigma_i \in \overline{\mathbb{K}}^{r\times r}$ diagonal s.t.

$$H_{\tilde{F}}^{\mathbf{b},x_i x_0^{-1}\mathbf{b}'} = U\,\Sigma_i\,V \quad \text{for } i \in 1{:}n$$

- ☞ $F = \sum_{i=1}^{r} w_i\, \xi_i^{\otimes d}$ with $\xi_i = ((\Sigma_j)_{i,i})_{j\in 1:n}$, $i \in 1{:}r$
- ☞ $U =$ **common eigenvectors** of $M_j := H_{\tilde{F}}^{\mathbf{b},x_j x_0^{-1}\mathbf{b}'} (H_{\tilde{F}}^{\mathbf{b},\mathbf{b}'})^{-1}$
  $=$ **evaluation** $\mathbf{e}_{\xi_i|\mathbf{b}}$ at $\xi_i$ for $i \in 1{:}r$ (up to scaling)
- ☞ $V^{-1} =$ **common eigenvectors** of $M_j' := (H_{\tilde{F}}^{\mathbf{b},\mathbf{b}'})^{-1} H_{\tilde{F}}^{\mathbf{b},x_j x_0^{-1}\mathbf{b}'}$
  $=$ **interpolation polynomials** at $\xi_1, \dots, \xi_r$ (up to scaling)

Let $F \in S^4(\mathbb{C}^3)$ with $\mathrm{rank} F = 4$ with 4 generic points $\xi_1, \ldots, \xi_4$, e.g.

$$\begin{aligned} F =& x0^4 + (x0 + x_1)^4 + (x_0 + x_2)^4 + (x_0 + x_1 + x_2)^4 \\ =& 4x_0^4 + 8x_0^3 x_1 + 8x_0^3 x_2 + 12x_0^2 x_1^2 + 12x_0^2 x_1 x_2 + 12x_0^2 x_2^2 + 8x_0 x_1^2 + 12x_0 x_1^2 x_2 + 12x_0 x_1 x_2^2 \\ &+ 8x_0 x_2^3 + 2x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + 2x_2^4 \end{aligned}$$

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- Let $h_k = \mathrm{rank} H_F^{d-k,k}$ for $0 \leq k \leq 4$:

$$\mathbf{h} = [1, 3, 4, 3, 1]$$

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- $H_0 := H_F^{2,2}$ of rank 4:

$$H_F^{2,2} = \begin{bmatrix} 4 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}$$

- $F_2^{\perp} = \ker H_0 = \langle q_1, q_2 \rangle \subset S^2$, e.g.

$$q_1 = x_0x_1 - x_1^2, q_2 = x_0x_2 - x_2^2$$

and $V(q_1, q_2) = \{\xi_1, \xi_2, \xi_3, \xi_4\}$.

☞ We find an **extensor** $\tilde{F} \in S^5(\mathbb{C}^3)$ such that $x_0 \star \tilde{F} = F$ with

$$H := H_{\tilde{F}}^{2,3} = \left[ \begin{array}{cc} H_0 & H_1 \end{array} \right]$$

where $H_0 = H_F^{2,2}$, s.t. $\tilde{F}_3^{\perp} = \ker H \supset \langle q_1 \cdot (x_0, x_1, x_2) + q_2 \cdot (x_0, x_1, x_2) \rangle$:

$$R = \left[ \begin{array}{cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right] = \left[ \begin{array}{c} R_0 \\ \hline \mathrm{R}_1 \end{array} \right]$$

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☞ Deduce by eigenvector comp. from $H$ of rank 4, the decomposition with $\xi_1 = [1:0:0], \xi_2 = [1:1:0], \xi_3 = [1, 0, 1], \xi_4 = [1:1:1]$:

$$H = \left[\begin{array}{cccccc|cccc} 4 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \end{array}\right]$$

# Symmetric tensor of order $4$, rank $6$ in $3$ variables

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The tensor:

$$T = 79\,x_0x_1^3 + 56\,x_0^2x_2^2 + 49\,x_1^2x_2^2 + 4\,x_0x_1x_2^2 + 57\,x_0^3x_1.$$

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The $15 \times 15$ Hankel matrix:

| | $1$ | $x_1$ | $x_2$ | $x_1^2$ | $x_1x_2$ | $x_2^2$ | $x_1^3$ | $x_1^2x_2$ | $x_1x_2^2$ | $x_2^3$ | $x_1^4$ | $x_1^3x_2$ | $x_1^2x_2^2$ | $x_1x_2^3$ | $x_2^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $0$ | $\frac{57}{4}$ | $0$ | $0$ | $0$ | $\frac{28}{3}$ | $\frac{79}{4}$ | $0$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $\frac{49}{6}$ | $0$ | $0$ |
| $x_1$ | $\frac{57}{4}$ | $0$ | $0$ | $\frac{79}{4}$ | $0$ | $\frac{1}{3}$ | $0$ | $0$ | $\frac{49}{6}$ | $0$ | $h_{500}$ | $h_{410}$ | $h_{320}$ | $h_{230}$ | $h_{140}$ |
| $x_2$ | $0$ | $0$ | $\frac{28}{3}$ | $0$ | $\frac{1}{3}$ | $0$ | $0$ | $\frac{49}{6}$ | $0$ | $0$ | $h_{410}$ | $h_{320}$ | $h_{230}$ | $h_{140}$ | $h_{050}$ |
| $x_1^2$ | $0$ | $\frac{79}{4}$ | $0$ | $0$ | $0$ | $\frac{49}{6}$ | $h_{500}$ | $h_{410}$ | $h_{320}$ | $h_{230}$ | $h_{600}$ | $h_{510}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ |
| $x_1x_2$ | $0$ | $0$ | $\frac{1}{3}$ | $0$ | $\frac{49}{6}$ | $0$ | $h_{410}$ | $h_{320}$ | $h_{230}$ | $h_{140}$ | $h_{510}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ | $h_{150}$ |
| $x_2^2$ | $\frac{28}{3}$ | $\frac{1}{3}$ | $0$ | $\frac{49}{6}$ | $0$ | $0$ | $h_{320}$ | $h_{230}$ | $h_{140}$ | $h_{050}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ | $h_{150}$ | $h_{060}$ |
| $x_1^3$ | $\frac{79}{4}$ | $0$ | $0$ | $h_{500}$ | $h_{410}$ | $h_{320}$ | $h_{600}$ | $h_{510}$ | $h_{420}$ | $h_{330}$ | $h_{700}$ | $h_{610}$ | $h_{520}$ | $h_{430}$ | $h_{340}$ |
| $x_1^2x_2$ | $0$ | $0$ | $\frac{49}{6}$ | $h_{410}$ | $h_{320}$ | $h_{230}$ | $h_{510}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ | $h_{610}$ | $h_{520}$ | $h_{430}$ | $h_{340}$ | $h_{250}$ |
| $x_1x_2^2$ | $\frac{1}{3}$ | $\frac{49}{6}$ | $0$ | $h_{320}$ | $h_{230}$ | $h_{140}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ | $h_{150}$ | $h_{520}$ | $h_{430}$ | $h_{340}$ | $h_{250}$ | $h_{160}$ |
| $x_2^3$ | $0$ | $0$ | $0$ | $h_{230}$ | $h_{140}$ | $h_{050}$ | $h_{330}$ | $h_{240}$ | $h_{150}$ | $h_{060}$ | $h_{430}$ | $h_{340}$ | $h_{250}$ | $h_{160}$ | $h_{070}$ |
| $x_1^4$ | $0$ | $h_{500}$ | $h_{410}$ | $h_{600}$ | $h_{510}$ | $h_{420}$ | $h_{700}$ | $h_{610}$ | $h_{520}$ | $h_{430}$ | $h_{800}$ | $h_{710}$ | $h_{620}$ | $h_{530}$ | $h_{440}$ |
| $x_1^3x_2$ | $0$ | $h_{410}$ | $h_{320}$ | $h_{510}$ | $h_{420}$ | $h_{330}$ | $h_{610}$ | $h_{520}$ | $h_{430}$ | $h_{340}$ | $h_{710}$ | $h_{620}$ | $h_{530}$ | $h_{440}$ | $h_{350}$ |
| $x_1^2x_2^2$ | $\frac{49}{6}$ | $h_{320}$ | $h_{230}$ | $h_{420}$ | $h_{330}$ | $h_{240}$ | $h_{520}$ | $h_{430}$ | $h_{340}$ | $h_{250}$ | $h_{620}$ | $h_{530}$ | $h_{440}$ | $h_{350}$ | $h_{260}$ |
| $x_1x_2^3$ | $0$ | $h_{230}$ | $h_{140}$ | $h_{330}$ | $h_{240}$ | $h_{150}$ | $h_{430}$ | $h_{340}$ | $h_{250}$ | $h_{160}$ | $h_{530}$ | $h_{440}$ | $h_{350}$ | $h_{260}$ | $h_{170}$ |
| $x_2^4$ | $0$ | $h_{140}$ | $h_{050}$ | $h_{240}$ | $h_{150}$ | $h_{060}$ | $h_{340}$ | $h_{250}$ | $h_{160}$ | $h_{070}$ | $h_{440}$ | $h_{350}$ | $h_{260}$ | $h_{170}$ | $h_{080}$ |

▶ Extract a $(6 \times 6)$ principal minor of full rank:

$$\mathrm{H}_{\sigma}^{B} = \begin{bmatrix} 0 & \frac{57}{4} & 0 & 0 & 0 & \frac{28}{3} \\ \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \end{bmatrix}$$

The columns (and the rows) of the matrix correspond to the monomials $B = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$. The shifted matrix $\mathrm{H}_{x_1\cdot\sigma}^{B,B}$ is

$$\mathrm{H}_{x_1\cdot\sigma}^{B,B} = \mathrm{H}_{\sigma}^{B,x_1B} = \begin{bmatrix} \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{79}{4} & 0 & 0 & h_{500} & h_{410} & h_{320} \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \end{bmatrix}$$

The columns of the matrix correspond to the monomials of $x_1 \cdot B$.
Similarly,

$$\mathrm{H}_{x_2\cdot\sigma}^{B,B} = \mathrm{H}_{\sigma}^{B,x_2B} = \begin{bmatrix} 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \\ 0 & 0 & 0 & h_{230} & h_{140} & h_{050} \end{bmatrix}$$

- Solve the commutation equations the unknonws $h_{500}, h_{410}, h_{320}, h_{230}, h_{140}, h_{050}$

$$\mathtt{M}_{x_i}\mathtt{M}_{x_j} - \mathtt{M}_{x_j}\mathtt{M}_{x_i} = \mathtt{H}_{x_1\cdot\sigma}\,\mathtt{H}_{\sigma}^{-1}\,\mathtt{H}_{x_2\cdot\sigma}\,\mathtt{H}_{\sigma}^{-1} - \mathtt{H}_{x_2\cdot\sigma}\,\mathtt{H}_{\sigma}^{-1}\,\mathtt{H}_{x_1\cdot\sigma}\,\mathtt{H}_{\sigma}^{-1} = 0.$$

A solution of the system is:

$$h_{500} = 1, h_{410} = 2, h_{320} = 3, h_{230} = 1.5060, h_{140} = 4.960, h_{050} = 0.056.$$

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$$\begin{bmatrix} 1 \\ -0.830+1.593\,i \\ -0.326-0.0501\,i \\ -1.849-2.645\,i \\ 0.350-0.478\,i \\ 0.103+0.0327\,i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.830-1.593\,i \\ -0.326+0.050\,i \\ -1.849+2.645\,i \\ 0.350+0.478\,i \\ 0.103-0.032\,i \end{bmatrix}, \begin{bmatrix} 1 \\ 1.142 \\ 0.836 \\ 1.305 \\ 0.955 \\ 0.699 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.956 \\ -0.713 \\ 0.914 \\ -0.682 \\ 0.509 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838+0.130\,i \\ 0.060+0.736\,i \\ 0.686-0.219\,i \\ -0.147-0.610\,i \\ -0.539+0.089\,i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838-0.130\,i \\ 0.060-0.736\,i \\ 0.686+0.219\,i \\ -0.147+0.610\,i \\ -0.539-0.089\,i \end{bmatrix}.$$

As the coordinates of the eigenvectors correspond to the evaluations of $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, we can recover the values of $(x_1, x_2)$.

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As the coordinates of the eigenvectors correspond to the evaluations of $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, we can recover the values of $(x_1, x_2)$.

- Solve the over-constrained linear system to obtain the weights:

$$\begin{aligned}(0.517 + 0.044\,i)\,(x_0 - (0.830 - 1.593\,i)x_1 - (0.326 + 0.050\,i)x_2)^4 \\ +(0.517 - 0.044\,i)\,(x_0 - (0.830 + 1.593\,i)x_1 - (0.326 - 0.050\,i)x_2)^4 \\ +2.958\,(x_0 + (1.142)x_1 + 0.836x_2)^4 \\ +4.583\,(x_0 + (0.956)x_1 - 0.713x_2)^4 \\ -(4.288 + 1.119\,i)\,(x_0 - (0.838 - 0.130\,i)x_1 + (0.060 + 0.736\,i)x_2)^4 \\ -(4.288 - 1.119\,i)\,(x_0 - (0.838 + 0.130\,i)x_1 + (0.060 - 0.736\,i)x_2)^4\end{aligned}$$

# Dimension extensions of multilinear tensors

## Definition

- $\tilde{T} \in \mathbb{K}^{\tilde{n}_1} \otimes \mathbb{K}^{\tilde{n}_2} \otimes \mathbb{K}^{\tilde{n}_3}$ is a dimension-**extension** of $T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3}$ if $\tilde{T}_{i_1,i_2,i_3} = T_{i_1,i_2,i_3}$ for $i_1 \in 1{:}n_1$, $i_2 \in 1{:}n_2$, $i_3 \in 1{:}n_3$
- $\tilde{T}$ is a dimension-**extensor** of $T$ if $\operatorname{rank} \tilde{T} = \operatorname{rank} T$.

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## Theorem (Strassen'83)

*A tensor $T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3}$ is of rank $r > \max(n_i)$ iff there exists an* **extensor** *$\tilde{T} \in \mathbb{K}^{r} \otimes \mathbb{K}^{r} \otimes \mathbb{K}^{n_3}$ of $T$ s.t.*

$$\tilde{T}_{[i]} = \tilde{U}\,\Sigma_i\,\tilde{V}^t \text{ for } i \in 1{:}n_3,$$

*with $\tilde{U}, \tilde{V}$ invertible and $\Sigma_i$ diagonal.*

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Such an extensor $\tilde{T}$ of $T$ can be obtained by completing $U \in \mathbb{K}^{n_1 \times r}$, $V \in \mathbb{K}^{n_2 \times r}$ such that $T = \sum_i U_i \otimes V_i \otimes W_i$ by **generic** matrices:

$$\tilde{U} = \begin{bmatrix} U \\ U' \end{bmatrix} \quad \tilde{V} = \begin{bmatrix} V \\ V' \end{bmatrix}$$

# Dimension extensors for symmetric tensors

## Definition

- For $E = \mathbb{K}^n$, $E' = \mathbb{K}^{n'}$, $n' > n$, $\tilde{F} \in S^d(E')$ is a dimension-**extensor** of $F \in S^d(E)$ if there exists $\pi : E' \to E$ projection s.t. $F = \pi^{\otimes d}(\tilde{F})$
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- $\tilde{F}$ is a dimension-**extensor** of $F$ if $\mathrm{rank}\tilde{F} = \mathrm{rank}F$.

☞ $F(x_1, \ldots, x_n) = \tilde{F}(x_1, \ldots, x_n, 0, \ldots, 0)$

## Theorem

*$F \in S^d(E)$ is of rank $\leq r$ iff there exists a* **space extensor** *$\tilde{F} \in S^d(E')$ of $F$, $\mathbf{b} \subset S^k(E'), \mathbf{b}' \subset (x_0) \cap S^{d-k}(E')$, $U, V \in \overline{\mathbb{K}}^{r\times r}$ invertible and $\Sigma_i \in \overline{\mathbb{K}}^{r\times r}$ diagonal s.t.*

$$H_{\tilde{F}}^{\mathbf{b}, x_i' x_0^{-1} \mathbf{b}'} = U \Sigma_i V \quad i \in 1{:}n'$$

# The connection with the Hilbert scheme

| | | |
|---|---|---|
| $I$ ideal of $S = \mathbb{K}[\bar{\mathbf{x}}]$ | | $\mathcal{D} = I^{\perp} \subset S^*$, stable by $x_i \star \cdot$ |
| $I$ defines $r$ points | $\Leftrightarrow$ | $\mathcal{D}$ of dimension $r \in \mathrm{Grass}_r(S^*)$ |
| (with multiplicity) | | |

# The connection with the Hilbert scheme

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**Hilbert scheme**

$\mathrm{Hilb}^r(\mathbb{P}^n)$ = set of homogeneous saturated ideals $I \subset S = \mathbb{K}[x_0, \ldots, x_n]$ such that $I$ defines $r$ points, counted with mult. (i.e. $\mathcal{A} = R/\underline{I}$ of dimension $r$).

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$I \in \mathrm{Hilb}^r(\mathbb{P}^n)$ represented by $I_\rho^{\perp} \in \mathrm{Hilb}_\rho^r$ its orthogonal in degree $\rho \geq r-1$:

$$I_\rho^{\perp} \equiv \nu_1 \wedge \cdots \wedge \nu_r \in \mathrm{Gr}_r(S_\rho^{*})$$

such that $x_i \star I_{\rho+1}^{\perp} \subset I_\rho^{\perp}$ for $i \in 1{:}n$.

☞ $\mathrm{Hilb}^r(\mathbb{P}^n) \subset \mathrm{Gr}_r(S_\rho^{*}) \subset \mathbb{P}(\wedge^r S_\rho^{*})$ for $\rho \geq r-1$

## Points in $\mathbf{Gr}_r(S_m^*)$

$= \{U_1 \wedge \cdots \wedge U_r, U_i \in S_m^*\} \subset \mathbb{P}(\wedge^r S_m^*)$

$= \mathcal{M}^o(r, s_m)/_r$: $U \sim U'$ iff $\exists G \in_r$ s.t. $U = G\, U'$

$= \mathrm{St}(r, s_m)/\mathcal{O}_r$: $U, U' \in \mathrm{St}(r, s_m) = \{UU^t = I_r\}$ (Stiefel manifold)

$U \sim U'$ iff $\exists O \in \mathcal{O}_r$ s.t. $U = O\, U'$.

**Plücker coordinates**: $\Delta_{\mathbf{b}}(U) = \det\left(U_{[:,\mathbf{b}]}\right)$ for $\mathbf{b} = \{b_{i_1}, \ldots, b_{i_r}\}$ a r-tuple of a (monomial) basis $\{b_1, \ldots, b_{s_m}\}$ of $S_m$.

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**Points in *$Hilb_r^m$*:** For an ideal $I \subset S$ defining $r$ points (c.w.m.)

- $$U = I_m^\perp = \begin{bmatrix} \nu_1(b_1) & \cdots & \nu_1(b_{s_m}) \\ \vdots & & \vdots \\ \nu_r(b_1) & \cdots & \nu_r(b_{s_m}) \end{bmatrix} = \begin{bmatrix} \cdots \nu_1 \cdots \\ \vdots \\ \cdots \nu_r \cdots \end{bmatrix}$$

  where $\{\nu_1, \ldots, \nu_r\}$ is a basis of $I_m^\perp$.

- $\mathbf{b} \subset S_m$ is a basis of $S_m/I_m$ iff $\Delta_{\mathbf{b}}(U) \neq 0$.
- $\nu_i : p \to \frac{1}{\Delta_{\mathbf{b}}} \Delta_{b_1, \ldots, b_{i-1}, p, b_{i+1}, \ldots, b_r}$

  defining the **normal form** $p \mapsto N(p) := \sum_{i=1}^r \nu_i(p)\, \mathbf{b}_i \quad (\equiv p \mod I)$

# Local description of the Hilbert Scheme

A covering of $\mathrm{Hilb}^r(\mathbb{P}^n)$:

$$\mathrm{Hilb}^m_{\mathbf{b}} = \{I \in \mathrm{Hilb}^m_r \mid \mathbf{b} \text{ basis of } \mathcal{A} = \underline{S_m}/\underline{I_m} \; i.e.\; \Delta_{\mathbf{b}}(I_m^{\perp}) \neq 0\}$$

for $\mathbf{b}$ a set of r $x_0$-monomials connected.

**Theorem**

*$\mathrm{Hilb}^m_{\mathbf{b}}$ is isomorphic to the set of all $\nu = (\nu^{\alpha}_{\beta})_{x^{\alpha}\in\partial\mathbf{b}, x^{\beta}\in\mathbf{b}}$ such that the operators of multiplication $M_{x_i}(\nu) : b \in \langle \mathbf{b}\rangle \mapsto N(x_i b) \in \langle \mathbf{b}\rangle$ commute:*

$$M_{x_i}(\nu) \circ M_{x_j}(\nu) - M_{x_j}(\nu) \circ M_{x_i}(\nu) = 0$$

*where*

$$N(\mathbf{x}^{\alpha}) = \sum_{\beta\in\mathbf{b}} \nu^{\alpha}_{\beta}\, \mathbf{x}^{\beta} \text{ for } \mathbf{x}^{\alpha} \in \partial\mathbf{b} = \mathbf{b}^{+} \setminus \mathbf{b}, \quad N(\mathbf{x}^{\beta}) = \mathbf{x}^{\beta} \text{ for } \mathbf{x}^{\beta} \in \mathbf{b}$$

☞ $M_{x_i}(\nu) = U_{\mathbf{b}}^{-1} U_{x_i\cdot\mathbf{b}}$ where $U \in \mathbb{K}^{r\times s_m}$ is representing $I_m^{\perp}$

[Alonso-Brachat-M'10]

See also [Bertone-Cioffi-Lella-Marinaria-Roggero'11,13; Kreuzer-Robbianno'11;Lederer'11; Brachat-Lella-M-Roggero'14]

For $F = \sum_{i=1}^{r} w_i \xi_i^{\otimes d}$ and $\tilde{F}$ a **degree-$(m+m')$ extensor of rank $r$** ($\tilde{F}^* = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} \, \mathbf{e}_{\xi_{\mathbf{i}}}$), with $\omega_{\mathbf{i}} \in \mathbb{K} \setminus \{0\}$ and $\xi_{\mathbf{i}} \in \mathbb{K}^n$ distinct, $m, m' \geq r$.
Let $\mathcal{D} = \mathbf{I}_{\mathbf{m}}^{\perp} \in \mathrm{Hilb}_m^r$ be the element associated to $\tilde{F}$.

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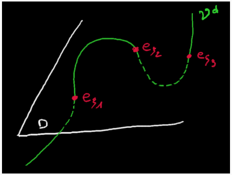
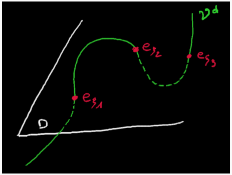



- $\mathcal{D} \equiv \mathbf{e}_{\xi_1}^{[m]} \wedge \cdots \wedge \mathbf{e}_{\xi_i}^{[m]}$ s.t. the evaluation $\mathbf{e}_{\xi}^{[m]}$ is a point of the Veronese variety $\mathcal{V}_n^m = \{(\xi^{\alpha})_{|\alpha|=m}, \xi \in \mathbb{C}^n\} \subset S_m^*$.

For $F = \sum_{i=1}^{r} w_i \xi_i^{\otimes d}$ and $\tilde{F}$ a **degree-$(m+m')$ extensor of rank $r$** $(\tilde{F}^* = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}}\, \mathbf{e}_{\xi_{\mathbf{i}}})$, with $\omega_{\mathbf{i}} \in \mathbb{K} \setminus \{0\}$ and $\xi_{\mathbf{i}} \in \mathbb{K}^n$ distinct, $m, m' \geq r$.
Let $\mathcal{D} = \mathbf{I}_{\mathbf{m}}^{\perp} \in \mathrm{Hilb}_m^r$ be the element associated to $\tilde{F}$.



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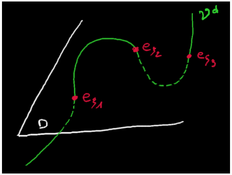

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- $\mathcal{D} \equiv U$ where $H_{\tilde{F}}^{m,m'} = U \Sigma^{[r]} V^t$ is the SVD decomposition with $U^t \in \mathrm{St}(r, s_m)$, $V \in \mathrm{St}(r, s_{m'})$, $s_k = \dim(S_k)$, $\Sigma^{[r]} = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$.

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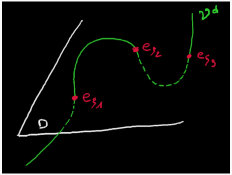


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- For $U \in \mathbb{K}^{r \times s_m}$ representing $\mathcal{D} \in \mathrm{Hilb}_m^r$ and $\mathbf{b}$ an $r$-tuple of $(x_0) \cap S_{m'}$ with $U_{\mathbf{b}}$ invertible,
  - ☞ The common eigenvectors of $[(U_{\mathbf{b}})^{-1} U_{x_i x_0^{-1} \mathbf{b}'}]_{i \in 1:n}$ are the **interpolation polynomials** at $\xi_i$ (up to scaling)
  - ☞ The common eigenvectors of $[U_{x_i x_0^{-1} \mathbf{b}} (U_{\mathbf{b}})^{-1}]_{i \in 1:n}$ are the **evaluations** $\mathbf{e}_{\xi_i}^{[\mathbf{b}]}$ (up to scaling)

# Connection with the variety of commuting matrices

**Definition (Commuting matrices)**

- $E = \mathbb{K}^r$
- $\mathcal{V}_{n,r} = \{D = (M_0, \ldots, M_n, w) \in (\mathrm{end(E)})^{n+1} \times E \mid M_i M_j - M_j M_i = 0\}$
- $D \in \mathcal{V}_{n,r}^{st}$ if there is no $S \subsetneq E$ such that $M_i(S) \subset S$, $w \in S$
- $\mathcal{M}_{n,r} = \mathcal{V}_{n,r}^{st}/\mathrm{Gl}_r$

**Property:** $\mathcal{M}_{n,r} \sim \mathrm{Hilb}_r(\mathbb{P}^n)$,
$D \in \mathcal{M}_{n,r} \mapsto I = \{p \in S \mid p(M_0, \ldots, M_n)\, w = 0\}$

$F \in S_d$ of rank $r \mapsto D = (M_0, \ldots, M_n, w) \in \mathcal{M}_{n,r}$ where $M_i = \tilde{H}_0^{-1} \tilde{H}_i$ and $w = x_0^m$.

Thanks for your attention