The symmetric geometric rank of symmetric tensors

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Joint work with Jose Rodriguez and Pierpaola Santarsierso

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Motivation

- There are many notions of the *rank* of a tensor
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- There are many notions of the *rank* of a tensor
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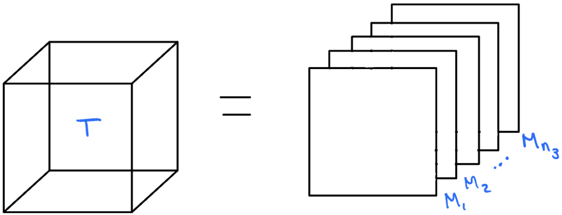
- This project is motivated by recent work of Kopparty, Moshkovitz and Zuiddam defining the *geometric rank* of a tensor [KMZ20]

Geometric rank

- Consider an order 3 tensor $T = (T_{ijk}) \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ with slices $M_1, \ldots M_{n_3}$ where $M_k = (T_{ijk})_{i,j} \in \mathbb{C}^{n_1 \times n_2}$, then the *geometric rank* of T is

$$\mathrm{GR}(T) = \mathrm{codim}\{(x, y) \in \mathbb{C}^{n_1 \times n_2} \ : x^T M_1 y = 0, \ldots, x^T M_{n_3} y = 0\}$$



Properties of the geometric rank

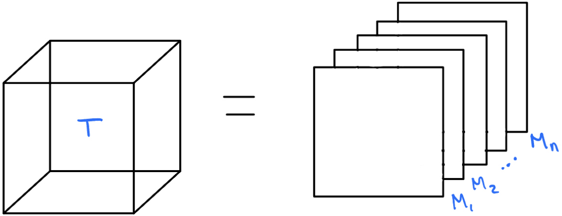
All of the following properties were proven in [KMZ20]:

1. GR is invariant under choice of slicing
2. If $S \leq T$ then $\mathrm{GR}(S) \leq \mathrm{GR}(T)$ where $S \leq T$ if $S = (A, B, C) \cdot T$
3. $\mathrm{GR}(S \oplus T) = \mathrm{GR}(S) + \mathrm{GR}(T)$
4. $\mathrm{GR}(S + T) \leq \mathrm{GR}(S) + \mathrm{GR}(T)$
5. $\mathrm{GR}(I_s) = s$
6. $\text{subrank}(T) \leq \text{border subrank}(T) \leq \mathrm{GR}(T) \leq \text{slice rank}(T)$
7. **Application:** The border subrank of $n \times n$ matrix multiplication is at most $\lceil \frac{3}{4} n^2 \rceil$

Symmetric geometric rank

- Given a symmetric tensor $T = (T_{ijk}) \in \mathrm{Sym}^3(\mathbb{C}^n)$ the *symmetric geometric rank* of T is

$$\mathrm{SGR}(T) = \mathrm{codim}\{x \in \mathbb{C}^n \ : \ x^T M_1 x = 0, \dots, x^T M_n x = 0\}$$



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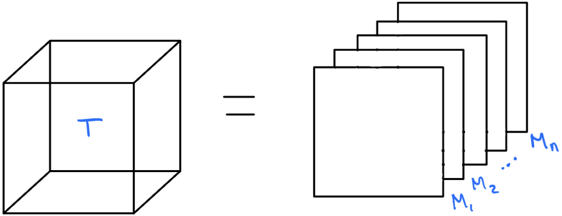

- $T = (T_{ijk}) \in \mathrm{Sym}^3(\mathbb{C}^n)$ can be seen as a homogeneous cubic polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$

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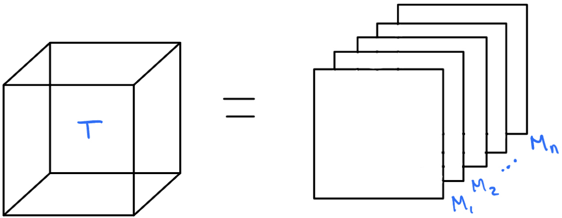

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- $0 \leq \mathrm{SGR}(T) \leq n$

Example

- Consider $T = [M_1 \ \ M_2]$ where

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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\mathrm{GR}(T) &= \operatorname{codim}\{(x,y) \in \mathbb{C}^2 \times \mathbb{C}^2 \;: x^T M_1 y = 0,\; x^T M_2 y = 0\} \\
&= \operatorname{codim}\{(x,y) \in \mathbb{C}^2 \times \mathbb{C}^2 \;: x_2 y_2 = 0,\; x_1 y_2 + x_2 y_1 = 0\} \\
&= 2 \\
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\end{aligned}$$

- The corresponding polynomial is $f = x_1 x_2^2$

Properties of symmetric geometric rank

We can prove analogous properties for symmetric geometric rank:

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$$\text{symmetric subrank}(T) \leq \mathrm{SGR}(T) \leq \mathrm{GR}(T) \leq \text{slice rank}(T)$$

Spaces of prescribed symmetric geometric rank

- The space of symmetric tensors $T \in \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^n))$ with SGR at most k is

$$\mathcal{S}_k = \{T \in \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^n)) \,|\, \mathsf{SGR}(T) \leq k\}.$$

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- $\mathcal{S}_{n-1}$ = singular polynomials $\Rightarrow$ defined by *discriminant*

Characterizing $\mathcal{S}_1$

- Let $N = \binom{n+d-1}{d} - 1$. The d-th Veronese embedding is

$$\begin{aligned} \nu_d : \mathbb{P}^{n-1} &\longrightarrow \mathbb{P}^N \\ [v] &\mapsto [v^{\otimes d}]. \end{aligned}$$

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Proposition (L., Rodriguez, Santarsiero, 2022)

The space of 3-ways symmetric tensors having symmetric geometric rank 1 is

$$\mathcal{S}_1 = \tau(X_3).$$

Characterizing $\mathcal{S}_2$

Theorem (L., Rodriguez, Santarsiero, 2022)

Any order 3 tensor in $\mathcal{S}_2$ is of the form

$$f = \ell_1^2 \cdot m_1 + \ell_2^2 \cdot m_2 + \ell_1 \ell_2 m_3$$

for linear forms $\ell_1, \ell_2, m_1, m_2, m_3$ or it is reducible.

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- Implies $\dim(\mathcal{S}_2) = \dim(\text{im}(\phi)) \leq 5(n-1)$ where

$$\phi : (\mathbb{P}^{n-1})^2 \times (\mathbb{P}^{n-1})^3 \mapsto \mathbb{P}^{\binom{n+2}{3}-1}$$
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- In general, can use this idea to get a lower bound $\dim(\mathcal{S}_k)$
  - Asymptotically tight as order of tensor goes to infinity [Sla15]

Facts about other $\mathcal{S}_k$

- Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. The *k-th secant variety* $\sigma_k(X)$ of X is

$$\sigma_k(X) = \overline{\bigcup_{p_1,\ldots,p_k \in X} \text{span}\{p_1, \ldots, p_k\}}.$$

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Let $\sigma_k(\tau(X_3))$ be the kth secant variety of the tangential variety $\tau(X_3)$ of the 3rd Veronese variety $X_3 \subset \mathbb{P}^{\binom{n+2}{3}-1}$. A general tensor in $\sigma_k(\tau(X_3)))$ has symmetric geometric rank k.

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- $\sigma_k(\tau(X_3))) \subseteq \mathcal{S}_k$ is strict unless $k = 1$

Future directions

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3. In progress: design algorithm to numerically compute $\text{GR}(T)$ and $\text{SGR}(T)$

References I

[KMZ20] Swastik Kopparty, Guy Moshkovitz, and Jeroen Zuiddam, *Geometric rank of tensors and subrank of matrix multiplication*, CoRR **abs/2002.09472** (2020).

[Sla15] Kaloyan Slavov, *The moduli space of hypersurfaces whose singular locus has high dimension*, Mathematische Zeitschrift **279** (2015), 139–162.