# Complexity of tensor decomposition and Hilbert functions

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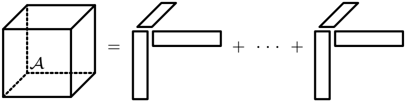
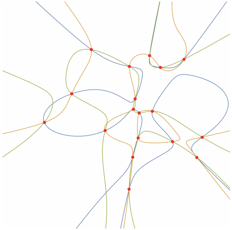
joint work with Nick Vannieuwenhoven and Fulvio Gesmundo



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# Tensor rank decomposition

Consider the vector space

$$\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \simeq \mathbb{C}^{(\ell+1)\times(m+1)\times(n+1)}$$

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This contains rank-1 tensors or elementary tensors

$$(\alpha \otimes \beta \otimes \gamma)_{ijk} := \alpha_i \beta_j \gamma_k$$

where $\alpha = (\alpha_i)_{0\leq i\leq \ell} \in \mathbb{C}^{\ell+1}$, $\beta \in \mathbb{C}^{m+1}, \gamma \in \mathbb{C}^{n+1}$.

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Every tensor $\mathcal{A}$ is a linear combination of rank-1 tensors:

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If $r$ is minimal, it is called the rank of $\mathcal{A}$ and the sum is called the tensor rank decomposition or canonical polyadic decomposition[1].

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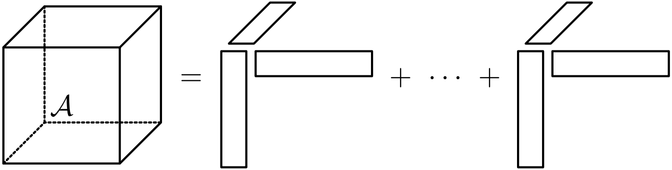
$$\mathcal{A} = \alpha_1 \otimes \beta_1 \otimes \gamma_1 + \cdots + \alpha_r \otimes \beta_r \otimes \gamma_r$$



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$$\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{\ell+1} \otimes (\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1})^{\vee}.$$

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The CPD $\mathcal{A} = \sum_{i=1}^r \alpha_i \otimes \beta_i \otimes \gamma_i$ gives

$$\begin{aligned}\mathcal{A}_{(1)} &= \alpha_1(\beta_1 \otimes \gamma_1)^\top + \cdots + \alpha_r(\beta_r \otimes \gamma_r)^\top \\ &= \begin{bmatrix} | & & | \\ \alpha_1 & \cdots & \alpha_r \\ | & & | \end{bmatrix} \begin{bmatrix} \text{———} & (\beta_1 \otimes \gamma_1)^\top & \text{———} \\ \text{———} & (\beta_2 \otimes \gamma_2)^\top & \text{———} \\ & \vdots & \\ \text{———} & (\beta_r \otimes \gamma_r)^\top & \text{———} \end{bmatrix}\end{aligned}$$

# Example

Consider the $4 \times 3 \times 3$ tensor $\mathcal{A}$ with flattening

$$\mathcal{A}_{(1)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 2 \end{bmatrix} \end{array}.$$

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$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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Question: how do we find $\alpha_i, \beta_i, \gamma_i$ from $\mathcal{A}_{(1)}$?

# From flattenings to polynomial equations

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$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array}
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Identify $\mathbb{C}^3 \otimes \mathbb{C}^3 \simeq$ bilinear forms in $\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}$.

$$f_1 = -x_1y_0 + x_1y_1, \quad f_2 = -x_0y_2 - x_1y_0 + x_1y_2, \quad f_3 = -2x_0y_0 + x_2y_0,$$
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The variety $V_{\mathbb{P}^2 \times \mathbb{P}^2}(f_1, \ldots, f_5)$ consists of the four points

$$(\beta_1, \gamma_1) = ((1:0:2),(1:0:0)), \quad (\beta_2, \gamma_2) = ((1:0:1),(0:1:0)),$$
$$(\beta_3, \gamma_3) = ((1:1:2),(0:0:1)), \quad (\beta_4, \gamma_4) = ((0:1:0),(1:1:1)).$$

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Notation: let $I = \langle f_1, \ldots, f_s \rangle \subset S = \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n]$. The degree of $f_i$ is $(1,1)$: $f_i \in S_{(1,1)}$.

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- $\mathcal{A}$ is $r$-identifiable, $\mathcal{A}_{(1)}$ has rank $r$ (Bocci, Chiantini, Ottaviani, '14)
- the ideal $I = \langle \ker \mathcal{A}_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle \subset S$ is such that $V_{\mathbb{P}^m \times \mathbb{P}^n}(I)$ is reduced and given by $\{(\beta_i, \gamma_i), i = 1, \ldots, r\}$ (Trisecant Lemma)

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A. Bernardi, J. Brachat, P. Comon, and B. Mourrain. "General tensor decomposition, moment matrices and applications". In: *J. Symbolic Comput.* 52 (2013), pp. 51–71

# Decomposing $\mathcal{A} \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$

Assumption: $\mathcal{A}$ is generic of rank $r \leq \min\{\ell + 1, mn\}$.

- $\mathcal{A}$ is $r$-identifiable, $\mathcal{A}_{(1)}$ has rank $r$ (Bocci, Chiantini, Ottaviani, '14)
- the ideal $I = \langle \ker \mathcal{A}_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle \subset S$ is such that $V_{\mathbb{P}^m \times \mathbb{P}^n}(I)$ is reduced and given by $\{(\beta_i, \gamma_i), i = 1, \ldots, r\}$ (Trisecant Lemma)

To find the $(\beta_i, \gamma_i)$, solve an overdetermined system of polynomial equations $f_1 = \cdots = f_s = 0$ on $\mathbb{P}^m \times \mathbb{P}^n$.


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# Hilbert functions and multiplication maps

# Hilbert functions and multiplication maps

For a graded $S$-module: $A = \bigoplus_{(d,e)\in\mathbb{Z}^2} A_{(d,e)}$, recall that the Hilbert function $\mathrm{HF}_A : \mathbb{Z}^2 \to \mathbb{N}$ is

$$\mathrm{HF}_A(d,e) = \dim_{\mathbb{C}} A_{(d,e)}.$$

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$$\mathrm{HF}_S(d,e) = \binom{m+d}{d}\binom{n+e}{e}, \ \mathrm{HF}_{S/I}(d,e) = \mathrm{HF}_S(d,e) - \mathrm{HF}_I(d,e)$$

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For $g \in S_{(d',e')}$, consider the linear multiplication map

$$M_g : (S/I)_{(1,1)} \to (S/I)_{(d,e)}$$

given by $M_g(f + I_{(1,1)}) = gf + I_{(d,e)}$, where $d = 1 + d'$, $e = 1 + e'$.

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# Solving via eigenvalues

By construction (recall $I = \langle \ker \mathcal{A}_{(1)} \rangle$, $\mathcal{A}_{(1)}$ has rank $r$)

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Lemma: If $h_0 \in S_{(d',e')}$ is such that $V_{\mathbb{P}^m \times \mathbb{P}^n}(h_0) \cap V_{\mathbb{P}^m \times \mathbb{P}^n}(I) = \emptyset$, $M_{h_0} : (S/I)_{(1,1)} \to (S/I)_{(d,e)}$ is invertible.

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Theorem: Let $(d, e), h_0$ be as above. The linear maps $M_{g/h_0} = M_{h_0}^{-1} \circ M_g$ for $g \in S_{(d',e')}$ commute and have eigenvalues

$$\frac{g}{h_0}(\beta_i, \gamma_i), \quad i = 1, \ldots, r.$$

# Computing multiplication matrices

- The matrices $M_{g/h_0}$ can be obtained from a basis of $I_{(d,e)}^{\perp}$ via the computation of a homogeneous normal form[2].

[2] S. T. "Solving Systems of Polynomial Equations (doctoral dissertation, KU Leuven)". In: (2020). available at https://simontelen.webnode.com/publications/.

# Computing multiplication matrices

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- For this approach to work, we need to find $(d', e') \neq 0$ such that $\mathrm{HF}_{S/I}(d, e) = r$. This is related to the regularity of $I$.

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Caveat: for very small ranks, $r \leq m + 1 \leq \ell + 1$, we can work with multiplication matrices $(S/I)_{(0,1)} \to (S/I)_{(1,1)}$ and our algorithm can be interpreted as a pencil-based algorithm.

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# Example (continued)

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\\2\end{bmatrix}\otimes\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\1\\2\end{bmatrix}\otimes\begin{bmatrix}0\\0\\1\end{bmatrix}+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\1\\1\end{bmatrix}$$

# Example (continued)

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\\2\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\\2\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$f_1 = -x_1 y_0 + x_1 y_1, \quad f_2 = -x_0 y_2 - x_1 y_0 + x_1 y_2, \quad f_3 = -2x_0 y_0 + x_2 y_0,$$
$$f_4 = -x_0 y_1 + x_2 y_1, \quad f_5 = -2x_0 y_2 + x_2 y_2.$$

# Example (continued)

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$\mathrm{HF}_{S/I}(i,j)$:

| $i$ \ $j$ | 0 | 1 | 2 | 3 | … |
|---|---|---|---|---|---|
| 0 | 1 | 3 | 6 | 10 | … |
| 1 | 3 | 4 | 4 | 4 | … |
| 2 | 6 | 4 | 4 | 4 | … |
| 3 | 10 | 4 | 4 | 4 | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

# Example (continued)

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| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

take $(d,e) = (2,1)$,
$h_0 = x_0 + x_1 + x_2$.

# Example (continued)

$$f_1 = -x_1 y_0 + x_1 y_1, \quad f_2 = -x_0 y_2 - x_1 y_0 + x_1 y_2, \quad f_3 = -2x_0 y_0 + x_2 y_0,$$
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| | $x_0^2y_0$ | $x_0^2y_1$ | $x_0^2y_2$ | $x_0x_1y_0$ | $x_0x_1y_1$ | $x_0x_1y_2$ | $x_0x_2y_0$ | $x_0x_2y_1$ | $x_0x_2y_2$ | $x_1^2y_0$ | $x_1^2y_1$ | $x_1^2y_2$ | $x_1x_2y_0$ | $x_1x_2y_1$ | $x_1x_2y_2$ | $x_2^2y_0$ | $x_2^2y_1$ | $x_2^2y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_0f_1$ | | | | −1 | 1 | | | | | | | | | | | | | |
| $x_1f_1$ | | | | | | | | | | −1 | 1 | | | | | | | |
| $x_2f_1$ | | | | | | | | | | | | | −1 | 1 | | | | |
| $x_0f_2$ | | | −1 | −1 | | 1 | | | | | | | | | | | | |
| $x_1f_2$ | | | | | | −1 | | | | −1 | | 1 | | | | | | |
| $x_2f_2$ | | | | | | | | | −1 | | | | −1 | | 1 | | | |
| $x_0f_3$ | −2 | | | | | | 1 | | | | | | | | | | | |
| $x_1f_3$ | | | | −2 | | | | | | | | | 1 | | | | | |
| $x_2f_3$ | | | | | | | −2 | | | | | | | | | 1 | | |
| $x_0f_4$ | | −1 | | | | | | 1 | | | | | | | | | | |
| $x_1f_4$ | | | | | −1 | | | | | | | | | 1 | | | | |
| $x_2f_4$ | | | | | | | | −1 | | | | | | | | | 1 | |
| $x_0f_5$ | | | −2 | | | | | | 1 | | | | | | | | | |
| $x_1f_5$ | | | | | | −2 | | | | | | | | | 1 | | | |
| $x_2f_5$ | | | | | | | | | −2 | | | | | | | | | 1 |

size $= \mathrm{HF}_S(d, e)$

# Example (continued)

We compute the matrices

$$M_{x_0/(x_0+x_1+x_2)}, M_{x_1/(x_0+x_1+x_2)}, M_{x_2/(x_0+x_1+x_2)}$$

from $I^{\perp}_{(2,1)}$.

# Example (continued)

We compute the matrices

$$M_{x_0/(x_0+x_1+x_2)}, M_{x_1/(x_0+x_1+x_2)}, M_{x_2/(x_0+x_1+x_2)}$$

from $I^{\perp}_{(2,1)}$. Their eigenvalues are the rows of

```
-1.03745e-16  0.25   0.333333       0.5
 1.0          0.25  -2.48091e-16  -3.16351e-16
-1.64372e-16  0.5    0.666667       0.5
```

# Example (continued)

We compute the matrices

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$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\\2\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\\2\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\1\\1\end{bmatrix}$$

# Example: $\mathcal{A} \in \mathbb{C}^{12} \otimes \mathbb{C}^7 \otimes \mathbb{C}^3$

For a generic tensor $\mathcal{A} \in \mathbb{C}^{12} \otimes \mathbb{C}^7 \otimes \mathbb{C}^3$ of rank 12, we find

$\mathrm{HF}_{S/I}(i,j)$ :

| $i$ \ $j$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 6 | 10 | 15 | 21 | ... |
| 1 | 7 | 12 | 15 | 16 | 15 | 12 | ... |
| 2 | 28 | 21 | 15 | 12 | 12 | 12 | ... |
| 3 | 84 | 12 | 12 | 12 | 12 | 12 | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

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|---|---|---|---|---|---|---|---|
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Here we can work with $(d, e) = (3, 1), (3, 2), (3, 3), (2, 3), \ldots$ but not with $(d, e) = (2, 1), (1, 2), (2, 2), \ldots$.

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Questions:

- Can we explain why $\mathrm{HF}_{S/I}(2,1) \neq 12$?

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| 2 | 28 | 21 | 15 | 12 | 12 | 12 | ... |
| 3 | 84 | 12 | 12 | 12 | 12 | 12 | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Here we can work with $(d,e) = (3,1),(3,2),(3,3),(2,3),\ldots$ but not with $(d,e) = (2,1),(1,2),(2,2),\ldots$.

Questions:

- Can we explain why $\mathrm{HF}_{S/I}(2,1) \neq 12$?
- Can we predict the first $d$ such that $\mathrm{HF}_{S/I}(d,1) = 12$?

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|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 6 | 10 | 15 | 21 | … |
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| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

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# Regularity

Let $(\beta_i, \gamma_i) \in \mathbb{P}^m \times \mathbb{P}^n$ be generic and

$$\mathcal{R}(m, n, (d, e)) := \frac{\mathrm{HF}_S(1,1)\mathrm{HF}_S(d', e') - \mathrm{HF}_S(d, e)}{\mathrm{HF}_S(d', e') - 1}.$$

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Conjecture: Let $I$ come from a tensor that satisfies our assumptions. Let $d \geq 1$ and let $m, n, r$ be such that

$$r \leq \mathcal{R}(m, n, (d, 1)).$$

There is a Zariski dense, open subset $U \subset (\mathbb{P}^m \times \mathbb{P}^n)^r$, such that for all $((\beta_i, \gamma_i), i = 1, \ldots, r) \in U$, $\mathrm{HF}_{S/I}(d, 1) = r$.

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$$r \leq \mathcal{R}(m, n, (d, 1)).$$

There is a Zariski dense, open subset $U \subset (\mathbb{P}^m \times \mathbb{P}^n)^r$, such that for all $((\beta_i, \gamma_i), i = 1, \ldots, r) \in U$, $\mathrm{HF}_{S/I}(d, 1) = r$.

Theorem: This conjecture holds in the following cases:

$$m \in \mathbb{N}_0,\ n \in \mathbb{N}_0 \ \&\ d = 1, \qquad m \in \{1, 2\},\ n \in \mathbb{N}_0 \ \&\ d = 2,$$
$$2 \leq m+1, n+1 \leq 50 \ \&\ d = 2, \qquad 2 \leq m+1, n+1 \leq 9 \ \&\ 3 \leq d \leq 5,$$
$$2 \leq m+1, n+1 \leq 6 \ \&\ 6 \leq d \leq 10.$$

# Decomposing a 7th order tensor of rank 1000

```
Grouping [[4, 6, 7, 8], [1, 2, 3], [5]] and reshaped to
(1050, 343, 6) tensor in 0.810859903 s
1. Performed ST-HOSVD compression to (1000, 343, 6) in 0.44888179 s
    Swapped factors 2 and 3, so the tensor has size (1000, 6, 343)
    Selected degree increment d_0 = [1, 0]
2. Constructed kernel of A_1 of size (1000, 2058) in 22.333814068 s
3. Constructed resultant map of size (7203, 6348) in 72.176802896 s
4. Constructed res res' in 1.266772414 s
5. Computed cokernel of size (1000, 7203) in 108.332858902 s
6. Constructed multiplication matrices in 2.037837294 s
7. Diagonalized multiplication matrices and extracted solution
    in 78.097176017 s
8. Refined factor matrices Y and Z in 151.170096114 s
9. Recovered factor matrix X in 0.186951757 s
10. Recovered the full factor matrices in 0.9396202759999999 s
Computed tensor rank decomposition in 440.457582263 s
Relative backward error = 3.873171296624731e-15
Relative forward error = 7.303893102189592e-14
```

# Decomposing a 7th order tensor of rank 1000

```
Grouping [[4, 6, 7, 8], [1, 2, 3], [5]] and reshaped to
(1050, 343, 6) tensor in 0.810859903 s
1. Performed ST-HOSVD compression to (1000, 343, 6) in 0.44888179 s
    Swapped factors 2 and 3, so the tensor has size (1000, 6, 343)
    Selected degree increment d_0 = [1, 0]
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Homotopy continuation would require tracking $> 40 \cdot 10^9$ paths!

# Points in projective space

Fix a degree $d \in \mathbb{N}$ (order of a partially symmetric tensor)

$Z = (z_1, \ldots, z_r) \in (\mathbb{P}^n)^r$ defines three ideals in $S = \mathbb{C}[x_0, \ldots, x_n]$:

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- $J(Z) = \langle f \in S \text{ homogeneous} : f(z_i) = 0 \text{ for all } i \rangle$,
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## Lemma

*If $r < \mathrm{HF}_S(d) - n$, then for $Z$ in a dense open subset $U \subset (\mathbb{P}^n)^r$*

1. $\mathrm{HF}_{J(Z)}(e) = \max\{\mathrm{HF}_S(e) - r, 0\}$,
2. $V_{\mathbb{P}^n}(I(Z, d)) = Z$ *(reduced)*,
3. $I(Z, d)^{\mathrm{sat}} = J(Z)$.

# Points in projective space

Q: *For* $\mathrm{HF}_S(d-1) < r < \mathrm{HF}_S(d) - n$ *and* $Z \in U$, *what is the smallest positive integer* $d'$ *such that* $\mathrm{HF}_{S/I(Z,d)}(d+d') = r$*?*

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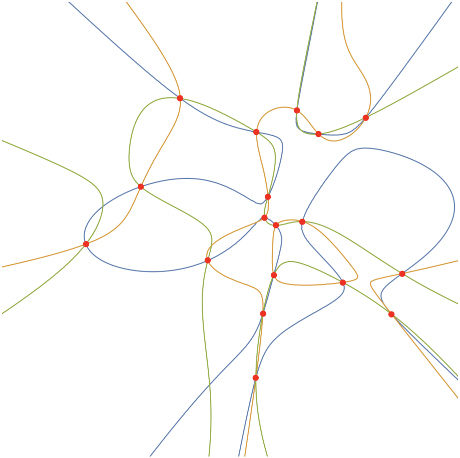

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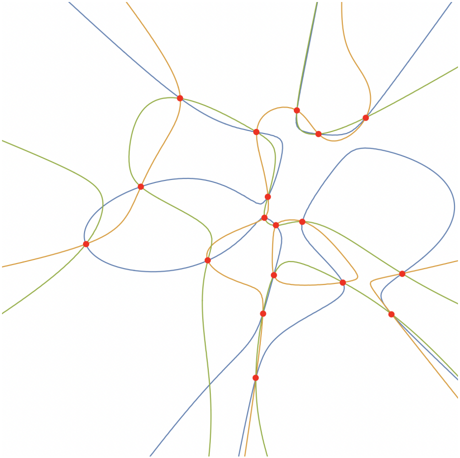

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The HF of $S/I(Z,5)$ is:

1 3 6 10 15 18 19 18 $\cdots$

A: $d' = 2$.

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A (Conjecture, T., Gesmundo): *Smallest* $d' > 0$ *such that*

$$(\mathrm{HF}_S(d') - 1) \cdot r \;\leq\; \mathrm{HF}_S(d') \cdot \mathrm{HF}_S(d) - \mathrm{HF}_S(d+d').$$

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Example: $n = 2, d = 5, r = 18$:

$$d' \neq 1: \quad 2 \cdot 18 > 3 \cdot 21 - 28$$
$$d' = 2: \quad 5 \cdot 18 = 6 \cdot 21 - 36$$

Example: $n = 3, d = 5$:

| $r$ | $\cdots$ | 46 | 47 | 48 | 49 | 50 | 51 | 52 |
|---|---|---|---|---|---|---|---|---|
| $d'$ | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 6 |

# Conclusion and outlook

- Under suitable assumptions, tensor rank decomposition is equivalent to computing points in $\mathbb{P}^m \times \mathbb{P}^n$ defined by a non-saturated, homogeneous ideal $I$, obtained from a flattening.

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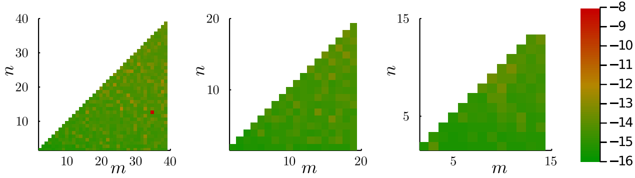
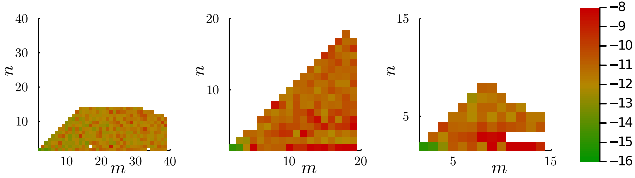
S.T. and N. Vannieuwenhoven, A Normal Form Algorithm for Tensor Rank Decomposition: arXiv:2103.07411 (to appear in ACM TOMS)

# Numerical experiments




(a) `cpd_hnf`




(b) `cpd_ddl`


I. Domanov and L. De Lathauwer. "Canonical polyadic decomposition of third-order tensors: relaxed uniqueness conditions and algebraic algorithm". In: *Linear Algebra Appl.* 513 (2017), pp. 342–375

I. Domanov and L. De Lathauwer. "Canonical polyadic decomposition of third-order tensors: Reduction to generalized eigenvalue decomposition". In: *SIAM J. Matrix Anal. Appl.* 35.2 (2014), pp. 636–660

# On our inequality

$$\begin{aligned}\mathrm{HF}_{S/I}(d+d') &= \mathrm{HF}_S(d+d') - \mathrm{HF}_I(d+d') \\ &\geq \mathrm{HF}_S(d+d') - \mathrm{HF}_S(d')\cdot(\mathrm{HF}_S(d)-r)\end{aligned}$$

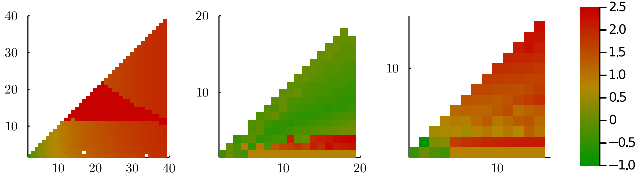
The inequality $\mathrm{HF}_S(d+d') - \mathrm{HF}_S(d')\cdot(\mathrm{HF}_S(d)-r) \leq r$ is equivalent to

A (Conjecture, T., Gesmundo): *Smallest $d' > 0$ such that*

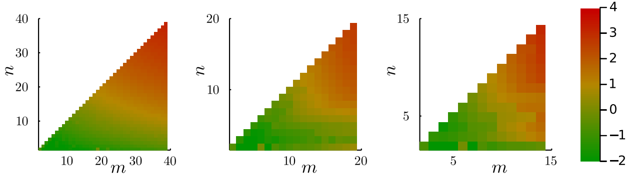
$$(\mathrm{HF}_S(d')-1)\cdot r \;\leq\; \mathrm{HF}_S(d')\cdot\mathrm{HF}_S(d) - \mathrm{HF}_S(d+d').$$

# Some more experimental data

Ratio of memory consumption:

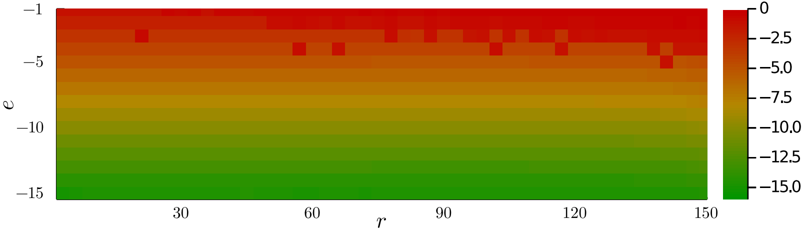



Computation time ($\log_{10}$, seconds)



# Some more experimental data

Low rank approximation with noise:




$(\ell, m, n) = (149, 24, 9)$, $e$ = noise level, $r$ = rank.