Hypergraph LSS-ideals and coordinate sections of symmetric tensors

Shekoofeh Gharakhloo

Joint work with Volkmar Welker

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AGATES, IMPAN, WARSAW

Outline

1. Introduction
   - History
2. Hypergraph LSS-ideals and coordinate sections of symmetric tensors
   - Algebraic properties of LSS–ideals for hypergraphs
   - Positive matching decomposition for hypergraph
3. References

- $\mathbb{K}$ field;
- $H = ([n], E)$ cluter hypergraph;
- $d \geq 1$ an integer;
- $S = \mathbb{K}[y_{ik} : \ i \in [n], \ k \in [d]]$;
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- $\mathbb{K} = \mathbb{R}$, H graph $\quad \Longrightarrow \quad V(L_H^{\mathbb{K}}(d)) = OR_{\bar{H}}(\mathbb{R}^d)$

An *orthogonal representation* of H in $\mathbb{R}^d$ assigns to each $i \in [n]$ a vector $u_i \in \mathbb{R}^d$ such that $u_i^T u_j = 0$, for $\{i, j\} \in \bar{E}$.

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- 1979 Lovász

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- 1979 Lovász
- 1984 Grötschel, Lovász and Schrijver

- $L_H^{\mathbb{K}}(d)$ Lovász-Saks-Schrijver ideal (LSS-ideal)

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- 2015 Herzog, Machia, Saeedi Madani and Welker




- 2018 Conca, Welker


Definition (Conca-Welker)

Given a hypergraph $H = (V, E)$, a *positive matching* of H is a matching M of H such that there exists a weight function $w\colon V \to \mathbb{R}$ satisfying:

$$\sum_{v \in e} w(v) > 0, \quad \text{if } e \in M,$$
$$\sum_{v \in e} w(v) < 0, \quad \text{if } e \notin M.$$

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Let $H = (V, E)$ be a hypergraph. A *positive matching decomposition* (or *pmd*) of H is a partition $E = \bigcup_{i=1}^{p} E_i$ of E into pairwise disjoint subsets such that E_i is a positive matching of $(V, E \setminus \bigcup_{j=1}^{i-1} E_j)$, for $i = 1, \ldots, p$. The E_is are called the *parts* of the pmd. The smallest p for which H admits a pmd with p parts will be denoted by $\mathrm{pmd}(H)$.

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The following theorem establishes a nice connection between the pmd of a hypergraph with the algebraic properties of the corresponding LSS-ideal.

Theorem (Conca-Welker)

Let $H = (V, E)$ be a hypergraph. Then for $d \geq \mathrm{pmd}(H)$ the ideal $L_H^{\mathbb{K}}(d)$ is a radical complete intersection. In particular, $L_H^{\mathbb{K}}(d)$ is prime if $d \geq \mathrm{pmd}(H) + 1$.

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The ideal $I_{d+1}^{\mathbb{K}}(Y_\Gamma^{\mathrm{Sym}})$ is radical (resp. is prime, has the expected height) if the ideal $L_\Gamma^{\mathbb{K}}(d)$ is radical (resp. is prime, is a complete intersection).

◁ LSS-ideal $L_\Gamma^{\mathbb{K}}(d)$ and determinantal ideal of the $(d+1)$-minors of a generic symmetric matrix $I_{d+1}^{\mathbb{K}}(Y_\Gamma^{\mathrm{Sym}})$:
The ideal $I_{d+1}^{\mathbb{K}}(Y_\Gamma^{\mathrm{Sym}})$ is radical (resp. is prime, has the expected height) if the ideal $L_\Gamma^{\mathbb{K}}(d)$ is radical (resp. is prime, is a complete intersection).
Therefore the ideal $I_{d+1}^{\mathbb{K}}(Y_\Gamma^{\mathrm{Sym}})$ is radical complete intersection for $d \geq \mathrm{pmd}(\Gamma)$ and is prime for $d \geq \mathrm{pmd}(\Gamma) + 1$.

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Let $\mathbb{K}$ be an algebraically closed field. Consider the map

$$\phi : (\mathbb{K}^n)^d \longrightarrow \underbrace{\mathbb{K}^n \otimes \ldots \otimes \mathbb{K}^n}_{k}$$

$$(v_1, \ldots, v_d) \mapsto \sum_{j=1}^{d} \underbrace{v_j \otimes \cdots \otimes v_j}_{k}$$

$$= \sum_{j=1}^{d} \sum_{1 \le i_1, \ldots, i_k \le n} (v_j)_{i_1} \cdots (v_j)_{i_k} \, e_{i_1} \otimes \ldots \otimes e_{i_k} \in \underbrace{\mathbb{K}^n \otimes \ldots \otimes \mathbb{K}^n}_{k}$$

The Zariski closure of the image of ϕ is the variety $S^d_{n,k}$ of symmetric tensors of (symmetric) rank $\leq d$. The coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_k}$ in $\phi(v_1, \ldots, v_d)$ is

$$\sum_{j=1}^{d} (v_j)_{i_1} \cdots (v_j)_{i_k} = f^{(d)}_{\{i_1 < \cdots < i_k\}}(v_1, \ldots, v_d).$$

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Therefore if we restrict the map ϕ to $V(L^{\mathbb{K}}_H(d))$, then we have a parameterization of the coordinate section of $S^d_{n,k}$ with 0 coefficient at $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for $\{i_1, \ldots, i_k\} \in E$.

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Theorem 1 (Gharakhloo-Welker)

Let $H = ([n], E)$ be a k-uniform hypergraph and $d \geq 2$. If $L_H^{\mathbb{K}}(d)$ is prime, then $L_H^{\mathbb{K}}(d)$ is a complete intersection.

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Theorem 2 (Gharakhloo-Welker)

Let $H = ([n], E)$ be a k-uniform hypergraph and $d \geq 2$. If $L_H^{\mathbb{K}}(d-1)$ is a complete intersection, then $L_H^{\mathbb{K}}(d)$ is prime.

Theorem (Conca-Welker)

Let $\mathbb{K}$ be a field. Then for a hypergraph $H = (V, E)$:

$$\mathrm{pmd}(H) \leq d \Rightarrow L_H^{\mathbb{K}}(d) \text{ is a complete intersection.} \quad (1)$$

Theorem (Avramov-Huneke)

Let R be a complete intersection and M be an R-module presented by the matrix $A \in R^{m\times n}$. Then

(1) $\mathrm{Sym}_R(M)$ is a complete intersection $\Leftrightarrow$ $\mathrm{height}\,(I_t(A)) \geq m-t+1$ for all $t = 1,\dots,m$.

(2) $\mathrm{Sym}_R(M)$ is a domain and $I_m(A) \neq 0$ $\Leftrightarrow$ R is a domain and $\mathrm{height}\,(I_t(A)) \geq m-t+2$ for all $t = 1,\dots,m$.

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For a k-uniform hypergraph $H=([n],E)$ and a number $d\geq 1$ we fix

$$S=\mathbb{K}\big[y_{ij} : i\in[n], j\in[d]\big], \qquad S'=\mathbb{K}\big[y_{ij} : i\in[n-1], j\in[d]\big],$$

$$H'=H\setminus\{n\}, \qquad R=S'/L^{\mathbb{K}}_{H'}(d),$$

$$U=\Big\{\{i_1,\dots,i_{k-1}\}\subseteq[n-1]\,\Big|\,\{i_1,\dots,i_{k-1},n\}\in E\Big\}, \qquad u=|U|.$$

Remark

Let $H = ([n], E)$ be a k-uniform hypergraph. Then $S/L_H^{\mathbb{K}}(d)$ is the symmetric algebra of the cokernel of the linear map $R^u \xrightarrow{A^T} R^d$ defined by the $u \times d$ matrix A where

$$A = \left(y_{i_1 j} y_{i_2 j} \cdots y_{i_{k-1} j}\right)_{\{i_1,\ldots,i_{k-1}\} \in U, j \in [d]} \in S'^{\,u \times d}.$$

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Lemma (Gharakhloo-Welker)

Let $H = ([n], E)$ be a k-uniform hypergraph. Then for every $2 \le t \le u$ the t-minor f_t of

$$A = (y_{i_1 j} y_{i_2 j} \cdots y_{i_{k-1} j})_{\{i_1,\ldots,i_{k-1}\}\in U, j\in[d]} \in S^{u\times d}$$

corresponding to the first t rows and columns is non-zero in $S/L_H^{\mathbb{K}}(d)$.

Definition (Gharakhloo-Welker)

For integers $k, c > 0$ such that $0 < k-1 \leq n-c$ let W be a set of $(k-1)$-subsets of $[n-c]$ with $|W| > 0$.
By $H_{W,c}$ we denote the hypergraph

$$\Big([n], \Big\{\{i_1,\dots,i_k\} \;\Big|\; \{i_1,\dots,i_{k-1}\} \in W,\ i_k \in \{n-c+1,\dots,n\}\Big\}\Big).$$

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Proposition (Gharakhloo-Welker)

If $L_H^{\mathbb{K}}(d)$ is prime, then H does not contain $H_{W,c}$ for any set W of $(k-1)$-subsets of $[n-c]$ with $|W| + c > d$.

Lemma (Gharakhloo-Welker)

For integers $n, u > 0$ *and* $k \geq 2$ *assume* $U = \{A_1, \ldots, A_u\}$ *for distinct* $(k-1)$*-subsets* $A_1, \ldots, A_u$ *of* $[n]$. *For elements* $(y_{ij})_{i\in[n], j\in[d]}$ *from the Noetherian ring* R *and variables* $(y_{i\,d+1})_{i\in[n]}$ *define* $m_{ij} = \prod_{\ell\in A_i} y_{\ell j}$ *for* $i \in [u]$ *and* $j \in [d]$. *Consider the matrix*

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1d} \\ m_{21} & m_{22} & \cdots & m_{2d} \\ \vdots & \vdots & & \vdots \\ m_{u1} & m_{u2} & \cdots & m_{ud} \end{bmatrix}$$

with entries in R and the matrix M' arising from M by adding the new column

$$[m_{1d+1}, \cdots, m_{ud+1}]^T$$

with entries in $T = R[Y] = R[y_{1d+1}, \cdots, y_{nd+1}]$.
Then for all $1 < t \leq u$ we have

$$\text{height } I_t(M') \geq \min\{\text{height } I_{t-1}(M), \text{height } I_t(M) + 1\}. \qquad (2)$$

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(T2) For each pair of vertices $v, v' \in V$, for which there is no edge in E containing both, there exists a unique sequence $e_1, \ldots, e_r \in E$ such that:

(a) $v \in e_1$ and $v' \in e_r$ and $v, v' \notin e_2, \ldots, e_{r-1}$,
(b) For each $1 \leq i \leq r-1$, we have $|e_i \cap e_{i+1}| = 1$,
(c) For each $1 \leq i \neq j \leq r$ where $|i - j| \geq 2$ we have $e_i \cap e_j = \varnothing$.

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◁ Let $k \geq 2$ and $H = (V, E)$ be a k-uniform hypergraph which is a tree. Then there exists at least one vertex $v \in V$, with $\deg_H(v) = 1$.

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◁ Let $k \geq 2$ and $H = (V, E)$ be a k-uniform hypergraph which is a tree. Then there exists at least one vertex $v \in V$, with $\deg_H(v) = 1$.

◁ Let $k \geq 2$ and let $H = (V, E)$ be a k-uniform hypergraph which is a tree with $\Delta(H) \geq 2$. Then there is a positive matching $M \subseteq E$ such that $V(M)$ contains all vertices v of degree $\deg(v) \geq 2$.

Corollary

Let $H = ([n], E)$ be a k-uniform tree. Then the coordinate sections of the variety $S^d_{n,k}$ with respect to H for $\Delta(H) + 1 \leq d \leq \binom{n+k-1}{k} - n$, are irreducible.

Proposition (Gharakhloo-Welker)

Let $H = (V, E)$ be the complete 3-uniform hypergraph on n vertices and with $\binom{n}{3}$ edges. Then for every $3 \leq l_1 \leq 2n-3$ and $5 \leq l_2 \leq 2n-1$, the set $E_{l_1,l_2} = \{\{a,b,c\} \in E \mid a < b < c, \quad a+b = l_1, b+c = l_2\}$ is a matching and $E = \bigcup_{l_1,l_2} E_{l_1,l_2}$.
In addition, the cardinality of the set

$$E_n := \{(l_1, l_2) \mid \text{ there exist } 1 \leq a < b < c \leq n, \quad l_1 = a+b, l_2 = b+c\}$$

is $\frac{3}{2}n^2 - \frac{15}{2}n + 10$.

Conjecture (Gharakhloo-Welker)

Let $H = (V, E)$ be a 3-uniform hypergraph with n vertices. Then $\mathrm{pmd}(H) \leq \frac{3}{2}n^2 - \frac{15}{2}n + 10$.

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◁ If the conjecture holds, then for $3\binom{n}{2} - 6n + 10 \leq d \leq \binom{n+2}{3} - n + 1$ every coordinate section of $S^d_{n,k}$ is irreducible.

A. Conca and V. Welker, *Lovász-Saks-Schrijver ideals and coordinate sections of determinantal varieties*, Algebra Number Theory **13**(2) (2019), 455–484.

M. Farrokhi D. G., Sh. Gharakhloo, and A. A. Yazdan Pour, *Positive matching decompositions of graphs*, Discrete Applied Mathematics, **320** (2022), 311-323.

Sh. Gharakhloo, V. Welker, *Hypergraph LSS-ideals and coordinate sections of symmetric tensors*, `arXiv:2202.10463`.

J. Herzog, A. Macchia, S. Saeedi Madani, and V. Welker, *On the ideal of orthogonal representations of a graph in* $\mathbb{R}^2$, Adv. Appl. Math. **71** (2015), 146–173.

Thanks for your attention