Set Optimization and Applications

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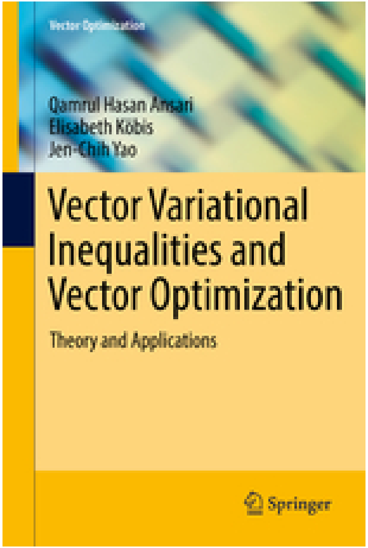
PART - I is extracted from the following book:



PART - II is based on the following chapter and papers:

- Ansari, Q.H., Sharma, P.K.: Set order relations, set optimization, and Ekeland's variational principle, In: V. Laha, P. Maréchal, S.K. Mishra, (eds.) Optimization, Variational Analysis and Applications, Springer Proceedings in Mathematics & Statistics, vol. 355, pp. 103-165, Springer Nature, Singapore Pvt. Ltd. (2021).
- Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209-236 (2011).
- Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Anal. **30**(3), 1487-1496 (1997).

Part - I

Vector Order

Vector Optimization

Vector Optimization

A vector optimization problem is stated as follows:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), f_2(x), \ldots, f_\ell(x)) \\ &\text{subject to } x \in K, \end{aligned} \tag{1}$$

where

- $X = \mathbb{R}^n$ is the *decision variable space*
- $\emptyset \neq K \subseteq \mathbb{R}^n$ is a *feasible region*
- $x = (x_1, x_2, \ldots, x_n)$ is the *decision (variable) vector*
- $f = (f_1, f_2, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^l$ is the *objective function*
- $f(K)$ is the *feasible objective region* and it is a subset of the *objective space* $Y = \mathbb{R}^l$.

Now, the question is, what does the word 'minimize' mean? Do we want to minimize all the objective functions simultaneously? If yes, then it can be achieved if there is no conflict between the objective functions. In this case, a solution can be found, without requiring any special method, where every objective function attains its minimum. Almost any real-world application of mathematics has conflictive multiple criteria.

For example, we consider a formal mathematical problem with feasible set $K = \{x \in \mathbb{R} : -5 \leq x \leq 5\}$ and objective functions

$$f_1(x) = x^2, \quad f_2(x) = x$$

and we want to minimize both objective functions over K:

$$\begin{aligned} &\text{minimize } (f_1(x), f_2(x)) \\ &\text{subject to } x \in K. \end{aligned}$$

In this problem, we have two criteria and one decision variable. Note that for each function individually the corresponding optimization problem is easy, and $x_1 = 0$ and $x_2 = -5$ are the (unique) minimizers for f_1 and f_2 on K, respectively. Now the questions are: What are the 'minima' and the 'minimizer' in this situation? Does $x_1 = 0$ or $x_2 = -5$ or both simultaneously minimize f_1 and f_2? Of course, the answer to the latter question is 'no'. We can see that there does not exist a solution that minimizes f_1 and f_2 at the same time.

For selecting a football team of 11 members among 50 players, we need to define a criteria, order or preference.

In general, to deal with multiobjective optimization problems, we need to define some criteria, order or preference.

Generally people consider the following ordering on the Euclidean space $\mathbb{R}^n$. For all $x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_\ell) \in \mathbb{R}^l$,

$$x \preccurlyeq y \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, \ell;\, x \neq y,$$

that is,

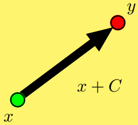
$$x \preccurlyeq y \;\Leftrightarrow\; x_i \leq y_i \text{ for all } i = 1, 2, \ldots, \ell \text{ and } x_j < y_j \text{ for some } j = 1, 2, \ldots, \ell.$$

This is equivalent to

$$x \preccurlyeq y \quad \Leftrightarrow \quad y - x \in C_0 \text{ (or } y \in x + C_0\text{)},$$

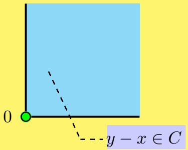
where $C_0 = \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.

Example $C := \mathbb{R}^2_+$

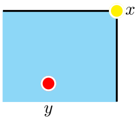


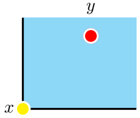
$$y = (0,3)$$
$$x = (-1,2)$$
$$y - x = (1,1) \geq (0,0)$$

$x \leq_C y$
$(y \geq_C x)$



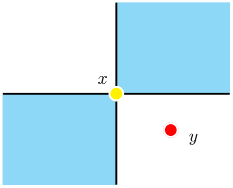
$y \in x - C \quad (y \leq_C x)$





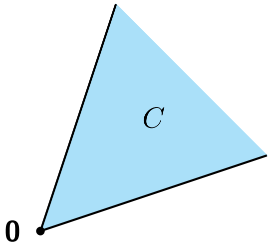
$y \in x + C \quad (x \leq_C y)$

In-comparable case



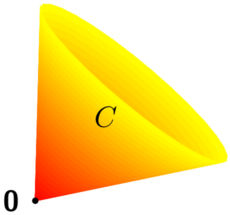
- A subset C of a vector space Y is said to be a cone if for all $x \in C$ and $\lambda \geq 0$, we have $\lambda y \in C$.
- The set C of Y is called a convex cone if it is convex and a cone; that is, for all $x, y \in C$ and $\lambda, \mu \geq 0$, we have $\lambda x + \mu y \in C$.
- A cone C in Y is said to be pointed if for $\mathbf{0} \neq x \in C$, we have $-x \notin C$, that is, $C \cap (-C) = \{\mathbf{0}\}$.

(a)



(a) Cone in $\mathbb{R}^2$.

(b)

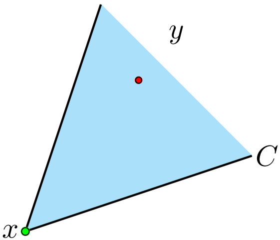


(b) Cone in $\mathbb{R}^3$.

Besides above ordering, we also have some other kinds of ordering on $\mathbb{R}^\ell$.

Name	Definition
weak componentwise order	if $x_i \leq y_i, \quad i = 1, 2, \ldots, \ell$
componentwise order	if $x_i \leq y_i, \quad i = 1, 2, \ldots, \ell;\ x \neq y$
strict componentwise order	if $x_i < y_i, \quad i = 1, 2, \ldots, \ell$
max order	if $\max\limits_{i=1,2,\ldots,\ell} x_i \leq \max\limits_{i=1,2,\ldots,\ell} y_i$

Therefore, in general, we define the ordering on $\mathbb{R}^n$ by a convex cone in the following way:



$$x \leq_C y \quad \Leftrightarrow \quad y \in x + C$$

$$(y - x \in C)$$

Proposition 1.1

Let C be a cone in a vector space X. Then $\leq_C$ defined as

$$x \leq_C y \quad \Leftrightarrow \quad y - x \in C \tag{2}$$

is compatible with scalar multiplication and addition in X, that is,

$$\text{for all } x, y \in X \text{ and } \lambda \geq 0 \quad : \quad x \leq_C y \ \Rightarrow \ \lambda x \leq_C \lambda y \tag{3}$$

and

$$\text{for all } x, y, z \in X \quad : \quad x \leq_C y \ \Rightarrow \ (x + z) \leq_C (y + z). \tag{4}$$

Furthermore,

(a) $\leq_C$ *is reflexive;*

(b) $\leq_C$ *is convex if and only if* $\leq_C$ *is transitive;*

(c) $\leq_C$ *is pointed if and only if* $\leq_C$ *is antisymmetric.*

Proposition Continue.....

Conversely, if $\preccurlyeq$ is a reflexive relation on X such that

$$\text{for all } x, y \in X \text{ and } \lambda \geq 0 \quad : \quad x \preccurlyeq y \;\Rightarrow\; \lambda x \preccurlyeq \lambda y \tag{5}$$

and

$$\text{for all } x, y, z \in X \quad : \quad x \preccurlyeq y \;\Rightarrow\; (x+z) \preccurlyeq (y+z), \tag{6}$$

then $C = \{x \in X : \mathbf{0} \preccurlyeq x\}$ is a cone and $\preccurlyeq$ and $\leq_C$ are equivalent.

(Strong) Efficient Element

Let A be a nonempty subset of a pre-ordered vector space Y with an ordering cone C.

- An element $\bar{y} \in A$ is called a *strong* (or *ideal*) *efficient element* or *strong* (or *ideal*) *minimal element* of the set A (with respect to C) if

$$A \subset \{\bar{y}\} + C, \quad \text{equivalently,} \quad \bar{y} \leq_C y \quad \text{for all } y \in A. \tag{7}$$

  We denote by $\mathbb{SE}(A, C)$ the set of all strong efficient elements of A with respect to C.

- An element $\bar{y} \in A$ is called a *strong* (or *ideal*) *maximal element* of the set A if

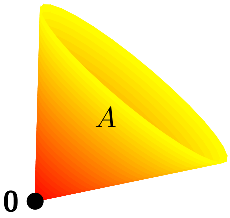
$$A \subset \{\bar{y}\} - C, \quad \text{equivalently,} \quad y \leq_C \bar{y} \quad \text{for all } y \in A. \tag{8}$$

(Strong) Efficient Element

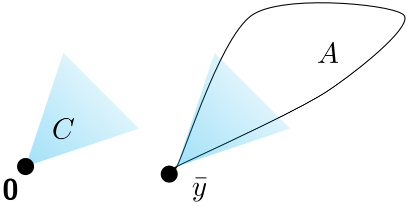
(a) The point $(0,0,0)$ is a strong efficient element of the set A, where the ordering cone is $C := \mathbb{R}^3_+$.

(b) The element $\bar{y}$ is a strong efficient element of the set A with the given ordering cone C.

(a)



(b)



Let A be a nonempty subset of a preordered vector space Y with an ordering cone C.

- An element $\bar{y} \in A$ is called an *efficient element* or *minimal element* of the set A (with respect to C) if
$$(\{\bar{y}\} - C) \cap A \subset \{\bar{y}\} + C. \tag{9}$$
The set of all efficient elements of A with respect to C is denoted by $\mathbb{E}(A, C)$.
- An element $\bar{y} \in A$ is called a *maximal element* of the set A if
$$(\{\bar{y}\} + C) \cap A \subset \{\bar{y}\} - C. \tag{10}$$

If the ordering cone C is pointed, then the inclusions (9) and (10) can be replaced, respectively, by the following relations:

$$(\{\bar{y}\} - C) \cap A = \{\bar{y}\}, \quad \text{equivalently,} \quad y \leq_C \bar{y},\ y \in A \Rightarrow y = \bar{y}. \tag{11}$$

and

$$(\{\bar{y}\} + C) \cap A = \{\bar{y}\}, \quad \text{equivalently,} \quad \bar{y} \leq_C y,\ y \in A \Rightarrow y = \bar{y}. \tag{12}$$

In other words, an element $\bar{y} \in A$ is said to be an *efficient element* or *minimal element* (respectively, *maximal element*) of the set A if there is no $y \in A$ with $y \neq \bar{y}$ and $y \leq_C \bar{y}$ (respectively, $\bar{y} \leq_C y$).

Illustration of an efficient element $\bar{y}$ of the set A.

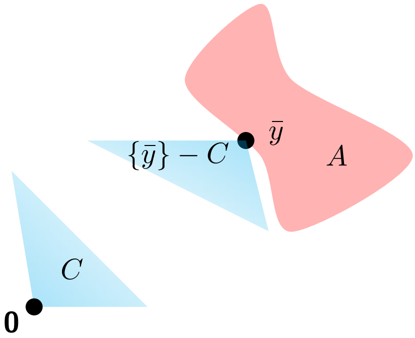
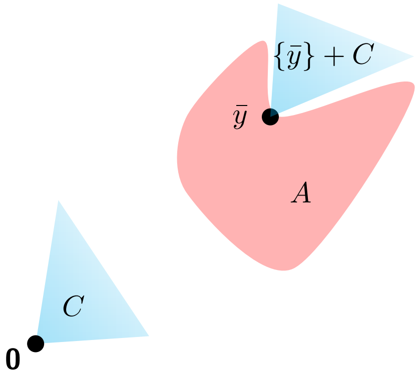


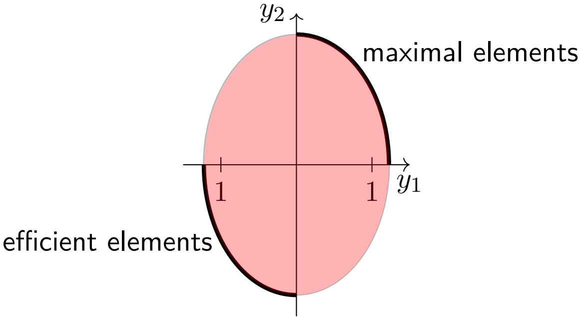
Illustration of a maximal element $\bar{y}$ of the set A.



Consider the set

$$A = \left\{(y_1, y_2) \in \mathbb{R}^2 : 2y_1^2 + y_2^2 \leq 3\right\}$$

with the ordering cone $C = \mathbb{R}^2_+$. Efficient and maximal elements of the set $A := \{(y_1, y_2) \in \mathbb{R}^2 : 2y_1^2 + y_2^2 \leq 3\}$, where the ordering cone is given by $\mathbb{R}^2_+$.

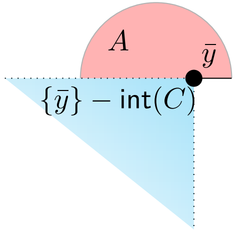


Weakly Efficient Element

Let A be a nonempty subset of a preordered vector space Y $(Y = \mathbb{R}^\ell)$ with an ordering cone C which has a nonempty interior.

- An element $\bar{y} \in A$ is called a *weakly efficient element* or *weakly minimal element* of the set A if
$$(\{\bar{y}\} - \text{int}(C)) \cap A = \emptyset. \tag{13}$$
The set of all weakly efficient elements of A with respect to C is denoted by $\mathbb{WE}(A, C)$.
- An element $\bar{y} \in A$ is called a *weakly maximal element* of the set K if
$$(\{\bar{y}\} + \text{int}(C)) \cap A = \emptyset. \tag{14}$$

A weakly efficient element $\bar{y}$ of a set A, where the ordering cone is given by $C = \mathbb{R}^2_+$.



Let C be a convex cone generating the preorder in Y. A point $\bar{x} \in K$ is said to be

- a *strongly efficient solution* of VOP if $f(\bar{x}) \in \mathbb{SE}(\mathcal{Y}, C)$;
- an *efficient* or *Pareto efficient solution* of VOP if $f(\bar{x}) \in \mathbb{E}(\mathcal{Y}, C)$;
- a *weakly efficient* or *weakly Pareto efficient solution* of VOP if $f(\bar{x}) \in \mathbb{WE}(\mathcal{Y}, C)$,

where $\mathcal{Y} = f(K)$.

Part - II

Set Order

Set Optimization

- Y : Vector space.
- $P(Y) := \{A \subseteq Y \ : \ A \neq \emptyset\}$.
- Let $A, B \in P(Y)$. The sum of A and B is defined as

$$A + B := \{a + b : a \in A,\ b \in B\},$$

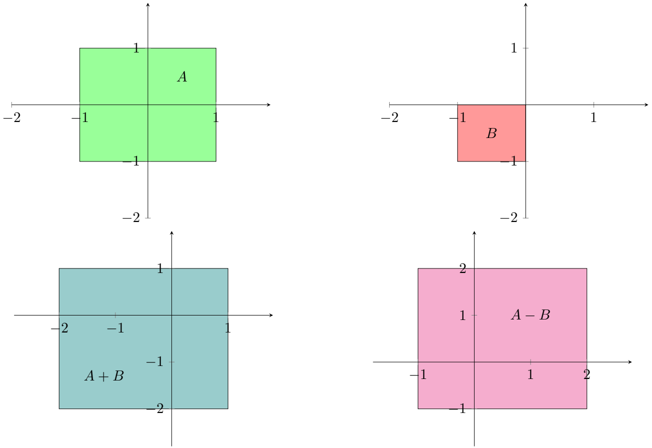
and the difference of A and B is defined as

$$A - B := \{a - b : a \in A,\ b \in B\}.$$

- Let X and Y be two nonempty sets. A set-valued map or multivalued map or point-to-set map or multifunction $F : X \rightrightarrows Y$ from X to Y is a map that associates with any $x \to X$ a subset $F(x)$ of Y; the set $F(x)$ is called the image of x under F.

- For $A = [-1,1] \times [-1,1]$ and $B = [-1,0] \times [-1,0]$, we have

$$A + B = [-2,1] \times [-2,1] \quad \text{and} \quad A - B = [-1,2] \times [-1,2].$$



Set Optimization Problem

- $\emptyset \neq S \subseteq X$: real vector space.
- Y : real topological vector space
- $F : S \rightrightarrows Y$ be a set-valued map with $F(x) \neq \emptyset$

The set-valued optimization problem (in short, SOP) is defined as:

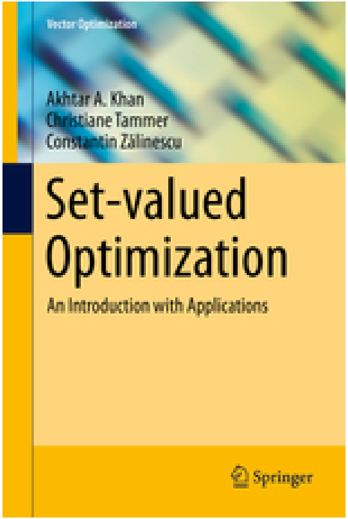
$$\begin{aligned} &\min \; F(x) \\ &\text{subject to } x \in S. \end{aligned} \tag{SOP}$$

The question is: what is the meaning of "min"?

There are two types of optimal solutions for set optimization problem in the literature:

- Vector approach : In vector approach one directly generalizes the concepts known from vector optimization to set optimization, that is, optimal solutions defined by vector orders, that is, find the efficient solutions of the set $F(S) := \bigcup_{x \in S} F(S)$.
- Set approach : In set approach optimal solutions are based on the set ordering defined on the power set.

For detail study on set optimization, we refer the following book.



- To find a footballer from a set of football players who has the least (or most) number of goals is the scalar optimization problem where the objective function gives the number of goals of a player.
- To find the footballer from a set of football players in such a way that he is having several qualities, namely, ability, speed, power, stamina, skill, popularity and so on, is a vector optimization problem. The value of the objective function can be regarded as vector whose coordinates consist of one's ability, speed, power, stamina, skill, popularity and so on.
- Consider the objective function whose values are teams and assume that a team is a set of football players and each player is regarded as a vector whose coordinates consist of one's ability, speed, power, stamina, skill, popularity and so on. Then one can formulate the problem of choosing a good team for a football league in the form of set optimization problem with the set-valued objective function.

Let $A, B \in P(Y)$ and C be a proper convex cone in Y. The set order relations on $P(Y)$ with respect to C are defined as follows:

- The lower set less order relation $\preceq_C^l$ is defined by

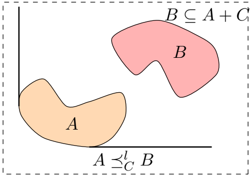
$$A \preceq_C^l B \quad \Leftrightarrow \quad B \subseteq A + C,$$

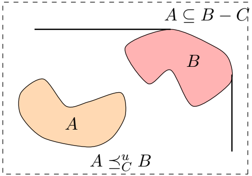
  or equivalently, for all $b \in B$, there exists $a \in A$ such that $a \preceq_C b$.

- The upper set less order relation $\preceq_C^u$ is defined by

$$A \preceq_C^u B \quad \Leftrightarrow \quad A \subseteq B - C,$$

  or equivalently, for all $a \in A$, there exists $b \in B$ such that $a \preceq_C b$.





Suppose you want to go to a restaurant for dinner, and all what counts for you is the quality of the meals. Suppose further that there are two restaurants in town, and by looking at their menus A and B on your smartphone you realize that for each meal on A there is one on B which you like better (or at least as much as the one on A). In which restaurant would you reserve a dinner table?

Formally, the relation between the two menus can be written as

$$\text{for all } a \in A, \text{ there exists } b \in B \text{ such that } a \leq_C b \tag{15}$$

where $\leq_C$ indicates your preference for (single) meals which is assumed to be a reflexive and transitive relation on the set of meals on any restaurant menu. In Kreps [1], this preference relation was also assumed to be total. This dinner decision process has a two stage character. In the first stage one chooses a menu (i.e., a restaurant), and in the second stage, only later, a dish from the chosen menu. Such a relation is nowadays called a set relation.

Kreps, D.M.: A representation theorem for "preference for flexibility". *Econometrica*, **47**(3) (1979), 565–577

Solution Concepts

The set order relations $\preceq_C^l$ and $\preceq_C^u$ are pre-order, that is, reflexive and transitive relations. In general, the set order relations $\preceq_C^l$ and $\preceq_C^u$ are not antisymmetric.

- An element $\bar{x} \in S$ is said to be a minimal solution of the problem (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is a minimal element of image set $F(S)$, that is,

$$(\{\bar{y}\} - C) \cap F(S) = \{\bar{y}\}.$$

The set of minimal elements of the problem (SOP) is denoted by $\mathrm{Min}(F, S)$.

- Let $\alpha \in \{l, u\}$. An element $\bar{x} \in S$ is said to be a α-minimal solution of the problem (SOP) with respect to the set order relation $\preceq_C^\alpha$ if and only if

$$F(x) \preceq_C^\alpha F(\bar{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\bar{x}) \preceq_C^\alpha F(x).$$

The set of α-minimal elements of the problem (SOP) is denoted by $\alpha - \mathrm{Min}(F, S)$, where $\alpha \in \{l, u\}$.

Example 1

For $S = \{0, 1, 2\}$, consider $F(0) = \{0\}, F(1) = \{1\}, F(2) = [0, 2]$ and $C = \mathbb{R}_+$. We can see that $\bar{x} = 2$ with $\bar{y} = 0$ is a minimal element of (SOP) via vector approach, although F(0) would be the better set because $F(1), F(2) \subseteq F(0) + C$ and hence $F(0) \preceq^l_C F(1), F(2)$.

Example 2

Let $X = \mathbb{R}, S = [0,1], Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 = x^2, v > 0\}, & \text{if } x \neq -1, 0, \\ (-1/2, 1), & \text{if } x = -1, \\ (1/2, 1), & \text{if } x = 0. \end{cases}$$

$\mathrm{Min}(F,S) = \emptyset$ and $u - \mathrm{Min}(F,S) = \{-1\}$.

Example 3

Let $X = \mathbb{R}, S = [0,1], Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [[(2,2),(3,3)]], & \text{if } x = 0, \\ [[(0,0),(4,4)]], & \text{otherwise.} \end{cases}$$

where $[[(a,b),(c,d)]] = \{(y_1,y_2) : a \leq y_1 \leq c, b \leq y_2 \leq d\}$. $\mathrm{Min}(F,S) = (0,1]$ and $u - \mathrm{Min}(F,S) = \{0\}$.

Let $\mathcal{Q} \subseteq P(Y)$ is equipped with $\preceq_C^\alpha, \alpha \in \{l, u\}$ and denote

$$F(S) := \{F(x) : x \in S\} \subseteq \mathcal{Q}.$$

- We say that F is semicontinuous with respect to the set order relation $\preceq_C^\alpha$ at $\bar{x} \in S$ if and only if $F(\bar{x}) \npreceq_C^\alpha V$ for some $V \in \mathcal{Q}$, then there is a neighborhood U of $\bar{x}$ in S such that

$$F(x) \npreceq_C^\alpha V, \quad \text{for all } x \in U.$$

The following assertions are equivalent:

(a) F is semicontinuous with respect to the set order relation $\preceq_C^\alpha$ on S.

(b) The level set with respect to the set order relation $\preceq_C^\alpha$ of F at $F(\bar{x})$

$$\mathcal{L}_F(F(\bar{x})) = \{x \in S : F(x) \preceq_C^\alpha F(\bar{x})\}$$

is closed.

Existence of optimal solutions to the set optimization problem:

Theorem 4

Suppose that S be compact, F be semicontinuous with respect to the set order relation $\preceq_C^\alpha$ on S and take values on $\mathcal{Q}$. Then, the problem (SOP) *has an optimal solution with respect to the set order relation $\preceq_C^\alpha, \alpha \in \{l, u\}$.*

Example 5

Let $X = \mathbb{R}, S = [0, 1], Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ with the set order relation $\preceq_C^l$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \{(u, v) \in \mathbb{R}^2 \,:\, (u - x)^2 + (v - x)^2 \leq 1\}.$$

For any $x \in [0, 1]$, the level set $\mathcal{L}_F(F(x))$ is equal to the closed interval $[0, x]$. Therefore, F is semicontinuous with respect to the set order relation $\preceq_C^l$ on $[0, 1]$. Furthermore, $x = 0$ is a optimal solution of the problem (SOP) with respect to the set order relation $\preceq_C^l$ on S.

Application to Radiotherapy Treatment

- Cancer is the leading cause of death in the world, but it can be treated if it is diagonised in its early stages.
- Several options are available for treatment: surgery, radiation therapy, chematherapy, hormone therapy, or a combination of such methods.
- Radiotherapy is the treatment of cancerous and dysplastic tissues with ionizing radiation that damages the deoxyribonuclic acid (DNA). While non-cancerous cells are in, means that small amounts of DNA damage renders them incapable of reproducing.
- The aim of the radiation therapy is to provide a high probability cancerous cell control while minimizing radiation damage to surrounding normal tissue.
- In the recent years, with improvements in medical technology and computer hard and software, an advanced treatment planning approach, known as Intensity Modulated Radiation Therapy (IMRT) has been introduced (first by Brahme [1] in 1988).

A. Brahme, *Optimization of stationary and moving beam radiation therapy techniques*, Radiotherapy and Oncology, **12**, 129–140, (1988).

Application to Radiotherapy Treatment

- The basic idea of IMRT is to reduce the intensity of rays going through particularly sensitive critical structures and to increase the intensity of these rays seeing primarily the target volume. The problem of calculating those intensities based on dose prescription in the target volume and the sorrounding critical structures is called inverse planning. This problem is modeled as a multi-criteria optimization problem with an objective function depending on the specific goal that the treatment planner wants to achieve.
- In general, a level dose of radiation in the cancer organ should be closed to desired dose while it is absolutely necessary to avoid radiation in the organs out side the tumor (the critical organs) as much as possible.

M. Ehrgott, C. Güler, H. W. Hamacher and L. Shao *Mathematical optimization in intensity modulated radiation therapy*, Ann Oper Res **175**, 309–365 (2010).

Application to Radiotherapy Treatment

- In IMRT, one searches for an optimal treatment plan $x \in S$ for the irradiation of a tumor. Thereby, one aims at reducing the radiation dose delivered to the neighbored healthy organs, while destroying the tumor.
- This is modeled in general as a multi-objective optimization problem with an objective $f_i : S \to \mathbb{R}$ $(i \in \{1, \ldots, k\})$ for each healthy neighbored organ measuring its dose stress.
- For calculating the dose stress $f_i(x)$ in an organ i for a given plan x, a model of the patient's body is used. This model is based on a partition of the relevant area into voxels, which are assigned to the different tissues and organs.
- This leads to a dose level $f_i(x, p, m)$ for each choice of the model m and the parameter p, and for safety reasons, it is preferable to consider the sets

$$F(x) = \bigcup_{p,m} \{(f_1(x,p,m), f_2(x,p,m), \ldots, f_k(x,p,m))\}$$

for each irradiation plan x, that is, we need to solve a set optimization problem with a set-valued map $F : S \rightrightarrows \mathbb{R}^k$.

Application to Game Theory with Uncertainty

Consider the following vector-valued game with uncertainty given by:

$$G := (\Gamma, \{X_\alpha\}, \{f_\alpha\}, \mathcal{U})_{\alpha \in \Gamma},$$

where,

- $\Gamma := \{1, 2, \cdots, n\}$ is the set of n players;
- For each $\alpha \in \Gamma$, X_α denotes the set of strategies of the αth player, which is a nonempty subset of a Hausdorff topological vector space X;
- Set $M := \Pi_{\alpha \in \Gamma} X_\alpha$;
- For each $\alpha \in \Gamma$ and the set $\mathcal{U}$ of all uncertainties, $f_\alpha : M \times \mathcal{U} \to Y$ is a loss function of the αth player.

- Set $M_{-\alpha} := \Pi_{\beta \in \Gamma \setminus \{\alpha\}} X_\beta$.
- For each $\alpha \in \Gamma$, we define

$$x_{-\alpha} := \{x_1, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_n\} \in M_{-\alpha}, \quad \forall\, x = (x_1, \cdots, x_n) \in M.$$

- If for each $\alpha \in \Gamma$, $w_\alpha \in X_\alpha$, then we define

$$(w_\alpha, x_{-\alpha}) := \{x_1, \cdots, x_{\alpha-1}, w_\alpha, x_{\alpha+1}, \cdots, x_n\} \in M.$$

- $f_\alpha(x, \mathcal{U}) := \{f_\alpha(x, u) \ : \ u \in \mathcal{U}\}$: represent all possible realizations of the vector-valued loss function.

Definition 6

An element $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in M$ is said to be

(a) a robust Nash equilibrium for the game G if and only if for any $\alpha \in \Gamma$,

$$f_\alpha(x_\alpha, \bar{x}_{-\alpha}, \mathcal{U}) \preceq_K^u f_\alpha(\bar{x}, \mathcal{U}),\ x_\alpha \in X_\alpha \Rightarrow f_\alpha(\bar{x}, \mathcal{U}) \preceq_K^u f_\alpha(x_\alpha, \bar{x}_{-\alpha}, \mathcal{U});$$

(b) a weak robust Nash equilibrium for the game G if and only if for any $\alpha \in \Gamma$, there is no $x_\alpha \in X_\alpha$ such that $f_\alpha(x_\alpha, \bar{x}_{-\alpha}, \mathcal{U}) \prec_K^u f_\alpha(\bar{x}, \mathcal{U})$

We denote the set of robust Nash equilibrium and weak robust Nash equilibrium for the game G by $\mathrm{REff}(F, K)$ and $\mathrm{RWEff}(F, K)$, respectively. From [1, Lemma 3.1], we have $\mathrm{REff}(F, K) \subseteq \mathrm{RWEff}(F, K)$.

Crespi, G.P., Kuroiwa, D., Rocca, M.: Robust Nash equilibria in vector-valued games with uncertainty. Ann. Oper. Res. **289**, 185-193 (2020).

Definition 7

An element $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in M$ is a robust Nash or weak robust Nash equilibrium if and only if $\bar{x}_\alpha \in X_\alpha$ is a efficient or a weak efficient solution, respectively, of the following set optimization problem

$$\begin{aligned} &\text{Minimize } f_\alpha(x_\alpha, \bar{x}_{-\alpha}, \mathcal{U}) \\ &\text{subject to } x_\alpha \in X_\alpha, \end{aligned} \qquad (P_\alpha)$$

for each $\alpha \in \Gamma$.

Questions?

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Thanks!