TENSOR DECOMPOSITIONS AND CLASSICAL ALGEBRAIC GEOMETRY

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These notes are intended to be up-to-date compared to previous similar lecture notes, like [CGO14], including references from the last decade, but at the same time they offer only a quick overview on the subject without the ambition of giving a complete picture, for which we refer to more extensive manuscripts like [BCC$^+$18].

Structure of the lectures and keywords.

Lecture I. Secant varieties and ranks with respect to different algebraic varieties (tensor rank, Waring rank, partially-symmetric rank, simultaneous symmetric rank, Waring-Chow rank, sums of powers, strength, slice rank, skew-symmetric rank); border rank; general and maximal ranks.

Lecture II. Expected dimension of secant varieties, defective varieties, secant varieties of curves, first Terracini's Lemma, defective surfaces and Severi's Theorem, contact loci and second Terracini's Lemma, tangential contact loci, identifiability.

Lecture III. Fat points, defective Veronese varieties, Alexander-Hirschowitz Theorem, the case of plane curves, Cremona reductions.

Lecture IV. Degeneration techniques, Castelnuovo's Exact sequence, (differential) Horace method, residue and trace, Segre-Veronese varieties and multiprojective linear systems, multiproj-affine-projective method, case of products of $\mathbb{P}^1$'s, defectiveness of Segre-Veronese varieties.

Lecture V. Apolarity action and Apolarity Lemma, catalecticant method, Sylvester's Algorithm for binary forms, VSP, Waring rank of monomials, e-computable forms, border rank and cactus rank, tame and wild forms, Border Apolarity Lemma.

Lecture I

1.1. **Secant varieties and ranks.** We work over an algebraically closed field $\Bbbk$ of characteristic 0.

Definition 1.1.1. Let X, Y be projective varieties embedded in n-dimensional projective space $\mathbb{P}^n$. The **join variety** of X and Y is the Zariski-closure of the union of all lines spanned by a point of X and a point of Y. I.e.,

$$J(X,Y) = \overline{\bigcup_{x\in X, y\in Y} \langle x, y\rangle} \subseteq \mathbb{P}^n.$$

Definition 1.1.2. Let X be projective variety embedded in n-dimensional projective space $\mathbb{P}^n$. The **r-th secant variety** of X is the Zariski-closure of the union of all linear spaces spanned by r points of X. I.e.,

$$\sigma_r(X) = \overline{\bigcup_{p_1,\ldots,p_r\in X} \langle p_1, \ldots, p_r\rangle} \subseteq \mathbb{P}^n.$$

In other words, $\sigma_1(X) = X$ and $\sigma_{r+1}(X) = J(X, \sigma_r(X))$.

Remark 1.1.3. It is called *r-th abstract secant variety* of X the closure of the of the set of tuples $(x_1, \ldots, x_r, p)$ where the x_i's are points of X spanning an $(r-1)$-dimensional linear space and p of such linear space, i.e.,

$$\mathfrak{s}_r(X) = \overline{\{(x_1,\ldots,x_r,p) \ : \ p \in \langle x_1,\ldots,x_r\rangle \cong \mathbb{P}^{r-1}\}} \subseteq X^{\times r} \times \mathbb{P}^n.$$

If X is irreducible, then abstract secant varieties are irreducible. Hence, the r-secant variety is irreducible since it is equal to the closure of image of the abstract r-secant variety via the projection onto the last factor.

If X is reducible, then the r-th secant variety of X is equal to the union of all possible joins of secant varieties of the irreducible components of X. I.e., if $X = \bigcup_{i=1,\ldots,m} X_i$, then

$$\sigma_r(X) = \bigcup_{j_1+\ldots+j_m=r} J(\sigma_{j_1}(X_1), \ldots, \sigma_{j_m}(X_m)),$$

The notion of secant varieties is strictly linked to a notion of rank.

Definition 1.1.4. Let X be a projective variety embedded in $\mathbb{P}^n$. For every $p \in \mathbb{P}^n$, the **X-rank** of p is the smallest number of points on X whose linear span contains p. I.e.,

$$\mathrm{rk}_X(p) = \min\{r \ : \ \exists x_1,\ldots,x_r \in X \text{ s.t. } p \in \langle x_1,\ldots,x_r\rangle\}.$$

Therefore, we immediately get that

$$\sigma_r(X) = \overline{\{p\in\mathbb{P}^n \ : \ \mathrm{rk}_X(p) \leq r\}}.$$

Under this general framework are collected several notions of ranks that are studied in the literature. Here are some of them. General references are [Lan12, BCC+18].

Example 1.1.5 (Matrix rank)**.** Let $\mathbb{P}^{mn-1} = \mathbb{P}(\mathrm{Mat}_{m\times n}(\Bbbk))$ be the projective space of $m \times n$ matrices. For every matrix $M \in \mathrm{Mat}_{m\times n}(\Bbbk)$ the usual notion of rank corresponds to the X-rank of the point $[M] \in \mathbb{P}^{mn-1}$ with respect to the variety X of rank-one matrices. Indeed, the matrix M has rank at most r if and only if it can be written as a sum of r rank-one matrices.

Example 1.1.6 (Tensor rank)**.** As we mentioned in Example 1.1.5, the notion of rank of matrices corresponds to the notion of rank with respect to the variety of rank-one matrices in the projective space of matrices. This can be generalized to *tensors*. Let $V_1, \ldots, V_d$ be $\Bbbk$-vector spaces and consider the tensor product $V_1 \otimes \ldots \otimes V_d$. A **rank-one tensor**, or **decomposable tensor**, is an element of the form $v_1 \otimes \ldots \otimes v_d$ where $v_i \in V_i$ for $i = 1, \ldots, d$. The **tensor rank** of any tensor $T \in V_1 \otimes \ldots \otimes V_d$ is the minimal number of rank-one tensors needed to write T as their linear combination. I.e.,

$$\mathrm{rk}(T) = \min\left\{ r \ : \ \exists v_{ij} \in V_j, \lambda_i \in \Bbbk \text{ s.t. } T = \sum_{i=1}^r \lambda_i v_{i1} \otimes \ldots \otimes v_{id} \right\}.$$

From the algebraic geometry perspective, the space of rank-one tensors is classically called **Segre variety**. That is the image of the regular map

$$\begin{array}{cccc} \nu_{\mathbf{1}}: & \mathbb{P}V_1 \times \ldots \times \mathbb{P}V_d & \longrightarrow & \mathbb{P}(V_1\otimes\ldots\otimes V_d), \\ & ([v_1],\ldots,[v_d]) & \mapsto & [v_1\otimes\ldots\otimes v_d]. \end{array}$$

Fixed projective coordinates, that is

$$((a_{1,0}:\ldots:a_{1,n_1}),\ldots,(a_{d,0}:\ldots:a_{d,n_d})) \mapsto (\ldots : a_{1,i_1}a_{2,i_2}\cdots a_{d,i_d} : \ldots).$$

Remark 1.1.7. An expression of a tensor as linear combination of rank-one tensors has been reintroduced several times in the literature since Hitchcock [Hit27]. For this reason, it can be found with several names like *Canonical Polyadic* (CP) [CC70] or *Parallel Factor* (PARAFAC) [Har70] decomposition. For applications of such decompositions, we refer to [KB09, Lan12]. In these notes, we simply refer to it as *tensor rank decomposition.*

Example 1.1.8 (Symmetric tensor rank, a.k.a Waring rank)**.** Let V be a $\Bbbk$-vector space and consider the symmetric power S^dV. That is the subspace of *symmetric tensors* in the tensor product $V^{\otimes d}$, i.e., tensors that are invariant under any permutation of the indices. Then, we may restrict to tensor rank decompositions whose summands are themeselves symmetric. The **symmetric tensor rank** of a symmetric tensor $T \in S^dV$ is

$$\mathrm{rk}^{\mathrm{sym}}(T) = \left\{ r \ : \ \exists v_j \in V, \lambda_j \in \Bbbk \text{ s.t. } T = \sum_{i=1}^r \lambda_i v_i^{\otimes d} \right\}.$$

Symmetric tensors can be identified with homogeneous polynomials. Given any rank-one tensor $v_1 \otimes \ldots \otimes v_d \in V^{\otimes d}$, its *symmetrization* is

$$\mathrm{sym}(v_1 \otimes \ldots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)} \in S^dV.$$

Then, if $\{x_0, \ldots, x_n\}$ is a basis of V, the identification between the symmetric tensors S^dV and degree-d homogeneous polynomials in the x_i's is done by extending linearly the following identification: given a multi-index $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ of lenght $|\alpha| = \alpha_0 + \ldots + \alpha_n = d$,

$$x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \quad \longleftrightarrow \quad \mathrm{sym}(\underbrace{x_0 \otimes \ldots \otimes x_0}_{\alpha_0} \otimes \ldots \otimes \underbrace{x_n \otimes \ldots \otimes x_n}_{\alpha_n}).$$

Under this identification, *symmetric rank decompositions* for symmetric tensors corresponds to decompositions of homogeneous polynomials as sums of powers of linear forms, a.k.a. *Waring decompositions.* Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a degree-d homogeneous polynomial. The **Waring rank** of f is the smallest number of d-th powers of linear forms needed to write f as their linear combination. I.e.,

$$\mathrm{wrk}(f) = \min \left\{ r \ : \ \exists \lambda_i \in \Bbbk, \ell_i \in \Bbbk[x_0, \ldots, x_n], \deg(\ell_i) = 1 \text{ s.t. } f = \sum_{i=1}^r \lambda_i \ell_i^d \right\}.$$

We call **Waring decomposition** an additive decomposition of a degree-d homogeneous polynomial as sums of d-th powers of linear forms. From the algebraic geometry perspective, the space of rank-one symmetric tensors, or equivalently of powers of linear forms, is classically called **Veronese variety**. That is the image of the map

$$\begin{array}{cccc} \nu_d : & \mathbb{P}V & \longrightarrow & \mathbb{P}S^dV, \\ & ([v]) & \mapsto & [v^{\otimes d}], \end{array}$$

which, considering the monomial basis $\{\binom{d}{\alpha} x^\alpha \ : \ \alpha \in \mathbb{N}^{n+1}, |\alpha| = d\}$ is given by

$$(a_0 : \ldots : a_n) \mapsto (a_0^d : a_0^{d-1}a_1 : \ldots : a_0^{d-i_1-\ldots-i_n} a_1^{i_1} \cdots a_n^{i_n} : \ldots : a_n^d).$$

Remark 1.1.9. Given a symmetric tensor $T \in S^dV$ we have introduced two notions of rank. It is natural to ask what is the relation between them. Since a symmetric tensor rank decomposition is a particular type of tensor rank decomposition, we obviously have

$$\mathrm{rk}(T) \leq \mathrm{rk}^{\mathrm{sym}}(T). \tag{1.1}$$

In 2008, Comon asked whether equality always holds or if there are examples of symmetric tensors whose symmetric tensor rank is strictly larger than its tensor rank, see [Oed08]. From a geometric perspective, Comon's question can be rephrased as whether the following inclusion is always an equality or not,

$$\sigma_r(\nu_d(\mathbb{P}V)) \subseteq \sigma_r(\nu_{\mathbf{1}}(\mathbb{P}V^{\times d})) \cap \mathbb{P}S^dV.$$

This question became soon one of the leading problems for the community and it became referred to *Comon's Conjecture* that equality holds. The conjecture was successfully proved in several cases (see [CGLM08, Fri16, ZHQ16, Sei20]) until 2018 when Shitov provided a counterexample to it, see [Shi18]. Shitov's remarkable counterexample is a cubic symmetric tensor over the complex numbers in 800 variables whose tensor rank is 903 while its symmetric tensor rank is 904. Other large examples over the real number appeared in [Shi20, WS22] for quartic and sextic real symmetric tensors.

Example 1.1.10 (Partially-symmetric rank)**.** Between the tensor rank and the symmetric rank there is a spectrum of partially-symmetric notions of ranks. Let $V_1, \ldots, V_m$ be $\Bbbk$-vector spaces and let $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$. Let $S^{\mathbf{d}}(V_1, \ldots, V_m) = S^{d_1}V_1 \otimes \ldots \otimes S^{d_m}V_m$ be the space of partially-symmetric tensors. Given a tensor $T \in S^{\mathbf{d}}(V_1, \ldots, V_m)$, we may restrict to tensor rank decompositions whose summands are rank-one partially-symmetric tensors. I.e.,

$$\mathrm{rk}^{\mathbf{d}}(T) = \min \left\{ r \ : \ \exists v_{ij} \in V_j, \lambda_i \in \Bbbk \text{ s.t. } T = \sum_{i=1}^r \lambda_i v_{i1}^{\otimes d_1} \otimes \ldots \otimes v_{im}^{\otimes d_m} \right\}.$$

From the geometric point of view, that is the rank with respect to the **Segre-Veronese variety** given by the image of the map

$$\begin{array}{rlcl} \nu_{\mathbf{d}} & : \mathbb{P}V_1 \times \ldots \times \mathbb{P}V_m & \longrightarrow & \mathbb{P}S^{\mathbf{d}}(V_1, \ldots, V_m) \\ & ([v_1], \ldots, [v_m]) & \mapsto & [v_1^{\otimes d_1} \otimes \ldots \otimes v_m^{\otimes d_m}] \end{array}$$

namely, $\nu_{\mathbf{d}} = \nu_{\mathbf{1}} \circ (\nu_{d_1} \times \ldots \times \nu_{d_m})$.

Remark 1.1.11. We can clearly generalize Comon's question to the case of partially symmetric tensors. Let $T \in S^{\mathbf{d}}(V_1, \ldots, V_m)$ be a partially symmetric tensor and let $\mathbf{d}' = (d'_{1,1}, \ldots, d'_{1,n_1}, \ldots, d'_{m,1}, \ldots, d'_{m,n_m})$ be a *refinement* of $\mathbf{d}$, i.e., for every $i \in \{1, \ldots, m\}$ we have $d_i = d'_{i,1} + \ldots + d'_{i,n_i}$. Then,

$$\mathrm{rk}^{\mathbf{d}'}(T) \leq \mathrm{rk}^{\mathbf{d}}(T). \tag{1.2}$$

The failure of Comon's conjecture opens up the question on the relations within the whole spectrum of partially symmetric ranks of symmetric tensors, see [GOV19]. Namely, can (1.2) be strict for any $\mathbf{d}'$ refinement of $\mathbf{d}$?

Remark 1.1.12 (Simultaneous rank)**.** The particular case of $(1, d)$-partially symmetric tensors can be interpreted also as *simultaneous rank*. Let $\mathcal{F} = \{f_1, \ldots, f_s\} \in S^d V$. The **simultanous rank** of $\mathcal{F}$ is the smallest set of d-th powers of linear forms that can be used to represent all polynomials of $\mathcal{F}$, i.e.,

$$w\mathrm{rk}(\mathcal{F}) = \min\{r \ : \ \exists \ell_1, \ldots, \ell_r \in S^1(V), f \in \langle \ell_1^d, \ldots, \ell_r^d \rangle \ \forall f \in \mathcal{F}\}.$$

Now, let $\{x_1, \ldots, x_s\}$ be a basis of $\mathbb{k}^s$ and consider the partially-symmetric tensors $T = \sum_{j=1}^s x_j \otimes f_j \in \mathbb{k}^s \otimes S^d V$. Then, $w\mathrm{rk}(\mathcal{F}) = \mathrm{rk}^{(1,d)}(T)$. Assume that $w\mathrm{rk}(\mathcal{F}) = r$. Then, there exists $\{\ell_1, \ldots, \ell_r\}$ such that $f_j = \sum_{i=1}^r \lambda_{j,i} \ell_i^d$. Therefore,

$$T = \sum_{j=1}^s x_j \otimes \left(\sum_{i=1}^r \lambda_{j,i} \ell_i^d \right) = \sum_{i=1}^r \left(\sum_{j=1}^s \lambda_{j,i} e_j \right) \otimes \ell_i^d,$$

i.e., $\mathrm{rk}^{(1,d)}(T) \leq r$. Viceversa, assume $\mathrm{rk}^{(1,d)}(T) = r$. I.e.,

$$T = \sum_{j=1}^s x_j \otimes f_j = \sum_{i=1}^r a_i \otimes \ell_i^d.$$

Consider a basis $\{\xi_1, \ldots, \xi_s\}$ for $(\mathbb{k}^s)^\vee$ dual to the basis $\{x_1, \ldots, x_s\}$. Applying on both sides, we obtain Waring decompositions $f_j = \sum_{i=1}^r \xi_j(a_i) \ell_i^d$. Hence, $w\mathrm{rk}(\mathcal{F}) \leq r$. Hence, we conclude that $w\mathrm{rk}(\mathcal{F}) = \mathrm{rk}^{(1,d)}(T)$.

Note also that, if $f \in S^d V$ and $\{x_1, \ldots, x_n\}$ is a basis for V, then we can regard f as a partially-symmetric tensor by writing

$$f = \frac{1}{d} \sum_{i=1}^n x_i \otimes \partial_{x_i} f \in V \otimes S^{d-1} V.$$

Therefore, from the previous observation, we have that the partially-symmetric rank $\mathrm{rk}^{(1,d-1)}(f)$ corresponds to the simultaneous rank of its partial derivatives, see [GOV19].

Example 1.1.13 (Waring-Chow ranks)**.** Let $f \in S_d$ be a degree-d homogeneous polynomial. Let $\mathbf{d} = (d_1, \ldots, d_m)$ be a partition of d, i.e., $d_1 + \ldots + d_m = d$. Then, the **d-rank**, or the **Waring-Chow rank**, of f is

$$w\mathrm{rk}_{\mathbf{d}}(f) = \min \left\{ r \ : \ \exists \ell_{ij} \in S_1, \lambda_i \in \mathbb{k} \text{ s.t. } f = \sum_{i=1}^r \lambda_i \ell_{i1}^{d_1} \cdots \ell_{im}^{d_m} \right\}.$$

Geometrically, the Waring-Chow rank corresponds to the rank with respect to the linear projection of the Segre-Veronese variety $\nu_{\mathbf{d}}(\mathbb{P}V)^{\times m}$ into the space of symmetric tensors. See [CCGO17]. The case $\mathbf{d} = \mathbf{1} = (1, \ldots, 1)$ goes usually under the name of **Chow rank**, see [AB11, TV21]. Another particular case is $\mathbf{d} = (d-1, 1)$ which goes under the name of **tangential rank** since it corresponds to the rank with respect to the *tangential variety* of the Veronese variety, see [AV18]: indeed, the tangent space at a point $[\ell^d] \in \nu_d(\mathbb{P}V)$ is given by

$$T_{[\ell^d]} \nu_d(\mathbb{P}V) = \{[\ell^{d-1} m] \ : \ m \in S_1\}.$$

Example 1.1.14 (Strength and symmetric slice rank)**.** Let $S = \mathbb{k}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$ be the standard graded polynomial ring where S_d denotes the space of degree-d homogeneous polynomials. Let $f \in S_d$. The **strength** of f is the smallest number of reducible forms that are needed to write f as their linear combination. I.e.,

$$\mathrm{str}(f) = \min \left\{ r \ : \ \exists \lambda_i \in \mathbb{k}, h_i \in S_{d_i}, g_i \in S_{d-d_i} \text{ s.t. } f = \sum_{i=1}^r \lambda_i h_i g_i \right\}.$$

The **slice rank** of f is the smallest number of forms with a linear factor that are needed to write f as their linear combination. I.e.,

$$sl\mathrm{rk}(f) = \min \left\{ r \ : \ \exists \lambda_i \in \mathbb{k}, \ell_i \in S_1, g_i \in S_{d-1} \text{ s.t. } f = \sum_{i=1}^r \lambda_i \ell_i g_i \right\}.$$

From a geometric point of view, the strength corresponds to the rank of the variety X_{red} of reducible forms in $\mathbb{P}S_d$. Observe that X_{red} has several components, among them the component of forms having a linear factor X_1. The slice rank corresponds to the rank with respect to X_1. See [CGG+19, BBOV21]. The notion of strength has been introduced in [AH20a] where the authors use it to solve a famous conjecture by Stillman on the existence of a uniform bound, independent on the number of variables, for the projective dimension of a homogeneous ideal of a polynomial ring. Since then, there has been a growing literature in the commutative algebra community exploiting and investigating this notion, see e.g. [AH20b, ESS21, Erm21].

Example 1.1.15 (Waring-type decompositions: sums of powers of higher degree forms)**.** In [FOS12], the authors considered additive decompositions of forms as sums of powers of higher degree forms. Given $f \in S_{dk}$, them the **k-th rank of f** is

$$\mathrm{rk}_k(f) = \min\left\{ r \ : \ \exists g_1, \dots, g_r \in S_d \text{ s.t. } f = \sum_{i=1}^{r} \lambda_i g_i^k \right\}.$$

Partial results for low degrees or low number of variables can be found in [FOS12, LORS19, FODT22].

Example 1.1.16 (Skew-symmetric rank)**.** Let V be a $\Bbbk$-vector space and consider the exterior power $\bigwedge^d V$. Analogously to the symmetric case, if we have a skew-symmetric tensor $T \in \bigwedge^d V$, we may restrict to tensor rank decompositions whose summands are themselves skew-symmetric. The **skew-symmetric rank** of a skew-symmetric tensor $T \in \bigwedge^d V$ is

$$\mathrm{rk}^{\mathrm{skew}}(T) = \min\left\{ r \ : \ \exists v_{ij} \in V, \lambda_i \in \Bbbk \text{ s.t. } T = \sum_{i=1}^{r} \lambda_i v_{i1} \wedge \ldots \wedge v_{id} \right\}.$$

From the perspective of algebraic geometry, skew-symmetric rank correspond to the rank with respect to *Grassmannians* embedded via *Plücker embedding*. In particular, if we denote by $\mathrm{Gr}(d, V)$ the space of d-dimensional linear subspaces of V, then

$$\begin{array}{cccc} \iota : & \mathrm{Gr}(d, V) & \longrightarrow & \mathbb{P}\bigwedge^d V, \\ & W = \mathrm{span}\{w_1, \dots, w_d\} & \mapsto & w_1 \wedge \ldots \wedge w_d. \end{array}$$

In coordinates, if $\{e_1, \dots, e_n\}$ is a basis of V, then

$$\iota(W) = \left[\sum_{\substack{I=(i_1,\dots,i_d) \\ i_1<\ldots<i_d}} w_{i_1,\dots,i_d} e_{i_1} \wedge \ldots \wedge e_{i_d} \right] \in \mathbb{P} \bigwedge^d V$$

where $w_{i_1,\dots,i_d}$ is the determinant of the $d \times d$ submatrix obtained selecting the rows $\{i_1, \dots, i_d\}$ from the $n \times d$ matrix having the vectors w_i's as columns. See [Bor13, ABMM21].

1.2. **General rank and maximal rank.** By definition, we have a chain of inclusions

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \ldots \subseteq \sigma_r(X) \subseteq \sigma_{r+1}(X) \subseteq \ldots \subseteq \mathbb{P}^n. \tag{1.3}$$

Lemma 1.2.1. *Let X be projective variety in $\mathbb{P}^n$. If $\sigma_r(X) = \sigma_{r+1}(X)$, then $\sigma_r(X) = \langle X \rangle$. In particular, $X = \sigma_2(X)$ if and only if X is a linear space.*

Exercise 1.1. Prove Lemma 1.2.1. Hint:

(1) show that $\sigma_2(X) = X$ if and only if X is a linear space;
(2) show that if $\sigma_r(X) = \sigma_{r+1}(X)$ then $\sigma_r(X) = \sigma_{r+h}(X)$ for any $h > 0$.

As immediate consequence of Lemma 1.2.1, we have that if X is *non-degenerate*, i.e., it is not contained in any proper linear space, then the chain of inclusions in (1.3) is proper until it fills the ambient space. Therefore, it makes sense to study what is the rank of a *general* point.

Definition 1.2.2. Let $X \subseteq \mathbb{P}^n$ be a non-degenerate variety. The **general X-rank** is the rank occurring in a non-empty Zariski-dense subset of $\mathbb{P}^n$. Namely,

$$\mathrm{rk}_X^{\circ} = \min\left\{ r \ : \ \sigma_r(X) = \mathbb{P}^n \right\}.$$

It is natural to ask whether the general rank coincides with the *maximal rank*, denoted by $\mathrm{rk}_X^{\max}$.

By definition, the set of rank-r points is contained in the r-th secant variety, i.e.,

$$\{p \in \mathbb{P}^n \ : \ \mathrm{rk}_X(p) = r\} \subseteq \sigma_r(X). \tag{1.4}$$

In the case of matrices, see Example 1.1.5, by lower semicontinuity of the rank, the inclusion (1.4) is actually an equality and the maximal rank coincides with the general one. However, in general, this is not the case as we can see from the following example.

Example 1.2.3. Let $\mathbb{P}^d = \mathbb{P}(\Bbbk[x_0, x_1]_3)$ be the projective space of cubic binary forms. Consider the *rational normal curve* $\mathcal{C}_3$ given by the image of the Veronese embedding

$$\nu_d : \mathbb{P}^1 \to \mathbb{P}^d, [\ell] \mapsto [\ell^d].$$

Note that

$$x_0x_1^2 = \lim_{t\to 0} \frac{1}{3t}\left[x_0^3 - (x_0 - tx_1)^3\right].$$

However, it can be proved that xy^2 cannot be written as a sum of two cubes, but

$$x_0x_1^2 = -\frac{1}{3}x_0^3 + \frac{2}{3}(x_0 + x_1)^3 + \frac{2}{3}(x_0 - x_1)^3. \tag{1.5}$$

Therefore, $[x_0x_1^2] \in \sigma_2(\mathcal{C}_3)$ but $w\mathrm{rk}(x_0x_1^2) = 3$.

Exercise 1.2. Show that $w\mathrm{rk}(x_0x_1^2) > 2$.

In Remark 2.2.3, we will see that the general binary cubic has rank 2 and therefore Example 1.2.3 provides a first example in which the maximal rank is strictly larger than the general one.

The failure of semicontinuity of rank makes necessary to introduce a semicontinuos notion of rank.

Definition 1.2.4. Let X be a projective varity embedded in $\mathbb{P}^n$. For every point $p \in \mathbb{P}^n$, the **border X-rank** of p is the smallest rank r for which there exists a one-parameter family of rank-r points which tends to p, i.e.,

$$\underline{\mathrm{rk}}_X(p) = \min\left\{r \ : \ \exists\{q_t\}_{t\in(0,1]}, \mathrm{rk}_X(q_t) = r \text{ s.t. } \lim_{t\to 0} q_t = p\right\}.$$

Equivalently, the smallest r for which the point p belongs to the r-th secant variety of X, i.e.,

$$\underline{\mathrm{rk}}_X(p) = \min\{r \ : \ p \in \sigma_r(X)\}.$$

Clearly, by definition, the general rank and the general border rank coincide.

Remark 1.2.5. The search for examples of points having rank strictly higher than the general one is extremely difficult and they are known in very few cases. Even if largely expected, we lack of a general proof that the maximal rank is strictly larger than the general one in the case of tensor ranks or its symmetric, partially-symmetric and skew-symmetric analogues. See [BHMT18, BBV22] for general study of high rank loci. See [Seg42, Kle99, Jel14, DP15, BT16] for other computations of maximal ranks, especially in the case of Veronese varieties.

Remark 1.2.6. A very particular case is the one of strength of polynomials, see Example 1.1.14. Indeed, it is easy to see that in the case of slice rank, i.e., secant varieties of the variety of homogeneous polynomials, the (1.3) is an equality. It is easy to observe that a form $f \in \Bbbk[x_0, \ldots, x_n]$ admits a slice decomposition of length r if and only if the hypersurface $\{f = 0\}$ contains a linear subspace of codimension r. Hence, in the projective space $\mathbb{P}S_d$ of degree-d homogeneous polynomials, the set of forms having slice rank at most r is the image of the projection onto the first factor of an incidence variety inside $\mathbb{P}S_d \times \mathbb{G}r(n-r, n)$, where $\mathbb{G}r(k, n)$ is the Grassmannian of k-dimensional linear spaces in $\mathbb{P}^n$; see [Har, Example 12.5]. Thus, it is Zariski-closed. As a consequence, the general slice rank is equal to the maximal slice rank in S_d. Moreover, from the latter classical construction, it is possible to deduce the value of the maximal slice rank for forms of fixed degree and number of variables, see [Har, Example 12.5]. In [BBOV22], it was proved that the same does not hold for the set of forms of bounded strength. Nevertheless, the general strength in S_d coincides with the maximal strength (and to the maximal slice rank): indeed, it was shown that the largest irreducible component of $\sigma_r(X_{\mathrm{red}})$ is $\sigma_r(X_1)$.

We conclude with a general upper bound on the maximal rank which is at most twice the general rank. This is far from being optimal (in the few cases a better bound has been proved), but it holds with respect to any algebraic variety. We recall the proof since it is beautifully simple.

Theorem 1.2.7. [BT15, Theorem 1] *Let $X \subseteq \mathbb{P}^n$ be a non-degenerate variety. Then,*

$$\mathrm{rk}_X^{\max} \leq 2\mathrm{rk}_X^\circ.$$

Proof. Let $U \subseteq \mathbb{P}^n$ the Zariski-dense set of points having general X-rank, i.e., for every $p \in U$, $\mathrm{rk}_X(p) = \mathrm{rk}_X^\circ$. Then, for any other point $q \in \mathbb{P}^n$, the line $\langle p, q\rangle$ through the two points contains infinitely many points of U. In particular, q is the linear combination of two points of U. By subadditivity of the rank, $\mathrm{rk}_X(q) \leq 2\mathrm{rk}_X^\circ$. □

Lecture II

We first focus on the following question.

Given $X \subseteq \mathbb{P}^n$ non-degenerate. What is the general X-rank?

As we have seen in the last lecture, this is equivalent to ask which is the smallest r such that $\sigma_r(X) = \mathbb{P}^n$.

2.1. Expected dimension.

As we observed in Remark 1.1.3 when we recalled the definition of abstract secant variety natural parametrization of the s-th secant variety of $X \subseteq \mathbb{P}^n$ is

$$\begin{array}{ccc} X^{\times s} \times \mathbb{P}^{s-1} & \longrightarrow & \mathbb{P}^n, \\ (p_1, \ldots, p_s, (a_1 : \ldots : a_s)) & \mapsto & a_1 p_1 + \ldots + a_s p_s. \end{array} \tag{2.6}$$

Hence, we can immediately deduce that

$$\dim \sigma_s(X) \leq s \dim(X) + s - 1,$$

where the right hand-side is equal to the dimension of the abstract secant variety of X. We might expect that such parametrization is optimal, namely that, as long as it is numerically possible, the projection from the abstract secant variety onto to secant variety has finite-dimensional fibers. I.e., the *expected dimension* of $\sigma_s(X)$ is

$$\exp.\dim \sigma_s(X) = \min\{n, s\dim(X) + s - 1\}.$$

Hence, we might say that expected answer to our question, namely the the *expeceted general X-rank* is

$$\exp.\mathrm{rk}_X^\circ = \left\lceil \frac{n+1}{\dim(X)+1} \right\rceil.$$

However, the dimension is not always as expected as we can easily see from the following easy example.

Example 2.1.1. Consider the Segre variety $X = \nu_{1,1}(\mathbb{P}^2 \times \mathbb{P}^2) \subseteq \mathbb{P}^8$ that is the variety of rank-1 matrices in the space of 3×3 matrices. Now, since X is is 4-dimensional, we expect that $\sigma_2(X) = \mathbb{P}^8$ fills the ambient space. However, it is immediate to see that it is not the case: indeed, as mentioned in Example 1.1.5, $\sigma_2(X)$ is the space of rank-2 matrices which is a hypersurface defined by the vanishing of the determinant.

Therefore, it makes sense to give the following definition.

Definition 2.1.2. Let $X \subseteq \mathbb{P}^n$ be a non-degenerate projective variety. We say that it is ***s*-defective** if the dimension of the s-th secant variety is not the expected one, i.e., $\dim \sigma_s(X) < \exp.\dim \sigma_s(X)$. In that case, we call the difference $\delta_s(X) = \exp.\dim \sigma_s(X) - \dim \sigma_s(X)$ the **geometric defect**.

We say that X is **defective** if it is s-defective for some s.

The classification of defective varieties is a challenging problem that goes back to the classical algebraic geometry of late XIX century and beginning of XX century. It is a classical result by Palatini [Pal09] that if a secant variety is not a hypersurface then it is of at least codimension two in the following one: this has immediate corollary that there are no defective curves, see Section 2.2. The classification of surfaces was due to Severi [Sev01] who proved that the only defective surfaces are the Veronese surface of $\mathbb{P}^5$ (namely, the space of rank-two ternary quadrics) and cones, see Section 2.3. The classification of defective threefolds is due to Scorza [Sco08]. The case of fourfolds was treated again by Scorza in [Sco09, Sco60] and completed recently in the recent pre-print [CCR20].

Other results have been given for particular families of algebraic varieties. The most celebrated result is the complete classification in the case of Veronese varieties where Alexander and Hirschowitz proved in 1995 [AH95] that all the examples of defective cases that were classically known since the beginning of XX century were, indeed, the only ones. Other partial classifications in the cases of Segre and Segre-Veronese varieties has been proved. We will come back on this in Lecture III and Lecture IV .

2.2. Secant varieties of curves.

We recall here the case of curves which will be an immediate corollary of the following result due to Palatini.

Theorem 2.2.1 ([Pal09])**.** *Let $X \subseteq \mathbb{P}^n$ be a non-degenerate irreducible variety and assume that $\dim \sigma_{s+1}(X) = \dim \sigma_s(X) + 1$. Then, $\sigma_s(X)$ is a hypersurface and $\sigma_{s+1}(X) = \mathbb{P}^n$.*

Proof. Let $p \in X$ be a general point and $q \in \sigma_{s+1}(X)$ be a smooth point. Hence,

$$\sigma_s(X) \subsetneq J(p, \sigma_s(X)) \subseteq \sigma_{s+1}(X).$$

By assumption and irreducibility of $\sigma_{s+1}(X)$, $J(p, \sigma_s(X)) = \sigma_{s+1}(X)$ and consequently $q \in \langle p, z\rangle$ for some $z \in \sigma_s(X)$. Being a cone, we also get that $p \in T_q\sigma_{s+1}(X)$. By generality of $p \in X$, we conclude that $\langle X \rangle \subseteq T_q\sigma_{s+1}(X)$ and then, since the opposite inclusion is trivial, $\sigma_{s+1}(X) = \mathbb{P}^n$. Consequently, $\sigma_s(X)$ is a hypersurface. □

In other words, if X is non-degenerate and $\sigma_s(X)$ is not a hypersurface, then $\sigma_s(X)$ has codimension at least two in $\sigma_{s+1}(X)$. We are now ready to compute the dimension of all secant varieties of non-degenerate curves.

Corollary 2.2.2. *Let $X \subseteq \mathbb{P}^n$ be a non-degenerate curve. Then,* $\dim \sigma_s(X) = \min\{2s-1, n\}$.

Proof. From the parametrization (2.6), we have seen that

$$\dim \sigma_s(X) \leq s \dim(X) + s - 1 = 2s - 1.$$

By Theorem 2.2.1,

$$\dim \sigma_s(X) \geq \dim \sigma_{s-1}(X) + 2$$

unless $\sigma_{s-1}(X)$ is an hypersurface. By induction on s, we deduce that $\dim \sigma_s(X) \geq 2(s-1) - 1 + 2 = 2s - 1$ as claimed. Finally, if $\sigma_{s-1}(X)$ is an hypersurface then $2(s-1) - 1 = n - 1$, i.e., $2s - 1 = n + 1$, and indeed $\dim \sigma_s(X) = n$ as claimed. □

Remark 2.2.3. From Corollary 2.2.2, the general rank with respect to any non-degenerate curve $\mathcal{C} \subseteq \mathbb{P}^n$ is

$$\mathrm{rk}^\circ_{\mathcal{C}} = \left\lceil \frac{n+1}{2} \right\rceil.$$

In particular, since, the Waring rank of binary degree-d forms corresponds to the rank with respect to the Veronese embedding of $\mathbb{P}^1$, i.e., with respect to a degree-d *Rational Normal Curve* in $\mathbb{P}^d$, see Example 1.1.8, then we deduce that the general Waring rank of degree-d binary homogeneous polynomials is

$$\mathrm{wrk}^\circ(d,2) = \left\lceil \frac{d+1}{2} \right\rceil.$$

This can be regarded as a geometric proof of a result due to Sylvester about decompositions of binary forms as sums of powers of linear forms [Syl51]. We will come back on Sylvester's approach in Lecture IV.

2.3. **First Terracini's Lemma.** We explain here a general geometric approach to the study of general ranks with respect to any algebraic variety. A natural approach to the computation of dimensions of algebraic varieties is through *general tangent spaces*. A classical result by Terracini gives us the following very useful description of the general tangent space of a secant variety.

Lemma 2.3.1 (First Terracini's Lemma, [Ter11])**.** *Let $X \subseteq \mathbb{P}^n$. Let $p_1, \ldots, p_r$ be general points on X and q be a general point on their linear span $\langle p_1, \ldots, p_r \rangle$. Then,*

$$T_q \sigma_r(X) = \langle T_{p_1}(X), \ldots, T_{p_r}(X) \rangle.$$

Proof. Let $\dim(X) = d$. Consider $X(t) = X(t_1, \ldots, t_d) \in \mathbb{P}^n$ be a local parametrization of X and, for $j = 1, \ldots, d$, denote by $X_j(t)$ the j-th partial derivative of such parametrization. Assume that, for any $i = 1, \ldots, r$, $p_i = X(t^i)$ for some $t^i = (t^i_1, \ldots, t^i_d)$. Then, the linear space $\langle T_{p_1}(X), \ldots, T_{p_r}(X) \rangle$ is spanned by the rows of the $r(d+1) \times (n+1)$ matrix whose rows are

$$\begin{pmatrix} X(t^1) \\ X_1(t^1) \\ \vdots \\ X_d(t^1) \\ \vdots \\ X(t^r) \\ X_1(t^r) \\ \vdots \\ X_d(t^r) \end{pmatrix}. \tag{2.7}$$

Similarly, we can write a local parametrization of $\sigma_r(X)$ as

$$s(t^1_1, \ldots, t^1_d, \ldots, t^r_1, \ldots, t^r_d, \lambda_1, \ldots, \lambda_{r-1}) = \sum_{i=1}^{r-1} \lambda_i X(t^i) + X(t^r).$$

The tangent space is spanned by the rows of the $r(d+1) \times (n+1)$ matrix whose rows are

$$\begin{pmatrix} \sum_{i=1}^{r-1} \lambda_i X(t^i) + X(t^r) \\ \lambda_1 X_1(t^1) \\ \vdots \\ \lambda_1 X_d(t^1) \\ \vdots \\ \lambda_{r-1} X_1(t^{r-1}) \\ \vdots \\ \lambda_{r-1} X_d(t^{r-1}) \\ X_1(t^r) \\ \vdots \\ X_d(t^r) \\ X(t^1) \\ \vdots \\ X(t^{r-1}) \end{pmatrix} \tag{2.8}$$

Clearly, the two matrices (2.7) and (2.8) are related one to the other by simple rows combination and the claim follows. □

Recalling the Grassmann formula on dimensions of span of linear spaces, Terracini's Lemma is telling us that a variety X is r-defective if and only if the tangent spaces at r general points of X have unexpected intersections. Even this reduces the problem of computing dimensions of secant varieties to a linear algebra problem, the latter quickly becomes infeasible to be computationally approached when the dimensions increase. Therefore, in order to be effectively use Terracini's Lemma, it is necessary to have a good description and interpretation of the tangent spaces of X. In Lecture III, we will see this in the case of Veronese varieties, where the problem of studying intersection of tangent spaces can be translated into an interpolation problem.

For now, let us see some example of defective variety directly in terms of Terracini's Lemma. Observe that, by Theorem 2.2.1, there are no defective surfaces in $\mathbb{P}^n$ for $n \leq 4$. In $\mathbb{P}^5$, the Veronese surface $\nu_2(\mathbb{P}^2)$ is defective.

Example 2.3.2 (Defective Veronese surface)**.** Let $X = \nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ be the image of the 2-nd Veronese embedding of $\mathbb{P}^2$. Being a surface, the expected dimension of its 2-nd secant variety is

$$\exp.\dim \sigma_2(X) = \min\{5, 2\cdot 2+1\} = 5.$$

In other words, we expect that $\sigma_2(X)$ fills the ambient space. Recall that the Veronese embedding can be regarded as the space of pure powers in the space of homogeneous polynomials, see Example 1.1.8. In this particular case, if $S = \Bbbk[x_0, x_1, x_2] = \bigoplus_{d\geq 0} S_d$ is the standard graded ring of ternary forms, then

$$X = \{[\ell^2] \ : \ \ell \in S_1\} \subseteq \mathbb{P}S_2.$$

The tangent space to a point $[\ell^2] \in X$ is easily computed. Since, for every $\ell, m \in S_1$,

$$\frac{d}{dt}(\ell + tm)^2|_{t=0} = 2\ell m,$$

then, we have that

$$T_{[\ell^2]}X = \{[\ell m] \ : \ m \in S_1\}.$$

By Terracini's Lemma, if $[\ell_1^2], [\ell_2^2] \in X$ are general and $p \in \langle [\ell_1^2], [\ell_2^2]\rangle$ is general, then

$$T_p\sigma_2(X) = \langle T_{[\ell_1^2]}X, T_{[\ell_2^2]}X\rangle.$$

By Grassmann formula,

$$\dim T_p\sigma_2(X) = \dim T_{[\ell_1^2]}X + \dim T_{[\ell_2^2]}X - \dim\left(T_{[\ell_1^2]}X \cap T_{[\ell_2^2]}X\right).$$

Now, it is immediate to see that $T_{[\ell_1^2]}X \cap T_{[\ell_2^2]}X$ is not empty, as expected, but $T_{[\ell_1^2]}X \cap T_{[\ell_2^2]}X = [\ell_1\ell_2]$. Therefore, $\dim T_p\sigma_2(X) = 2+2+0 = 4 < 5$.

Example 2.3.3. Another easy example of defective varieties is given by cones. Indeed, every pair of tangent spaces meets at the vertex of the cone and, therefore, their span has dimension lower than the expected.

As mentioned above, an important result by Severi showed that the latter two examples are the only defective cases for surfaces [Sev01]. See e.g. [Chi04] for a complete exposition.

Theorem 2.3.4 (Severi's Theorem, [Sev01])**.** *Let $X \subseteq \mathbb{P}^n$ be a 2-defective surface. Then, $n \geq 5$ and X is either projectively equivalent to a cone or to the Veronese surface $\nu_2(\mathbb{P}^2)$.*

It is immediate to see that, with a similar idea as in Example 2.3.2, it is possible to show that indeed *all* degree-2 Veronese embedding are defective. We will see this in Example 3.2.1 by following a different approach.

Exercise 2.3. Let $X = \nu_{\mathbf{1}}(\mathbb{P}^m \times \mathbb{P}^n)$ be the Segre variety of rank-one matrices. Compute the dimension of the s-th secant variety $\sigma_s(X)$. For which s it is defective?

2.4. **Contact loci and second Terracini's Lemma.** We recall subvarieties useful to understand defective varieties. For a complete presentation, we refer to [Chi04].

Definition 2.4.1. For general points $\{p_1, \ldots, p_r\} \subset X$ and a general hyperplane $H \supset \langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle$, we call **contact locus** $\Sigma(H) = \Sigma_{p_1,\ldots,p_r}(H)$ the union of all the components of the set $\{q \in X^{\rm smooth} ~:~ T_q(X) \subset H\}$ containing one of the p_i's. If $\Sigma(H)$ is positive dimensional, then we say that X is **weakly defective**.

Lemma 2.4.2 (Terracini's Second Lemma, see [BO08, Lemma 2.3])**.** *Let $X \subset \mathbb{P}^n$ be a r-defective variety. Let $\{p_1, \ldots, p_r\}$ be a set of general points. A general hyperplane $H \subset \mathbb{P}^n$ which contains $\langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle$ is indeed tangent to X along a positive-dimensional variety C passing through the p_i's.*

Proof. Let $q \in \langle p_1, \ldots, p_r\rangle$ be a general point. We consider the projection of the abstract r-secant variety in $X^{\times r} \times \mathbb{P}^n$ (see Remark 1.1.3) onto the r-secant variety. Along this projection, q admits a positive dimensional fiber $\pi^{-1}(q)$ since, by assumption, X is r-defective. Note that for any $(x_1, \ldots, x_r) \in \pi^{-1}(q)$ we have that $q \in \langle x_1, \ldots, x_r\rangle$. In particular, for any such r-tuple, we have that
$$T_{x_1}X \subset \langle T_{x_1}(X), \ldots, T_{x_r}(X)\rangle = T_q(\sigma_r(X)) = \langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle.$$
Hence, the projection of $\pi^{-1}(q)$ to one of the first factor defines a subvariety $C \subset X$ which concludes the proof. $\square$

Definition 2.4.3. For a general set of points $\{p_1, \ldots, p_r\} \subseteq X$, consider the Zariski-closure of
$$\{q \in X^{\rm smooth} ~:~ T_q(X) \subseteq \langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle\}.$$
and let $\Gamma_{p_1,\ldots,p_r}$ be the union of the components containing at least one of the p_i's. We call it k-**tangential contact locus**. If the latter is positive dimensional, we say that X is k-**tangentially defective**.

Clearly, for general points $\{p_1, \ldots, p_r\} \subset X \subset \mathbb{P}^n$ and a general hyperplane $H \subset \mathbb{P}^n$, we have that
$$\Gamma_{p_1,\ldots,p_r} \subset \Sigma(H).$$

Lemma 2.4.4. *Let $X \subseteq \mathbb{P}^n$ be r-defective. Let $\{p_1, \ldots, p_r\} \subset X$ be general and $L = \langle T_{p_2}(X), \ldots, T_{p_r}(X)\rangle$. Let π_L the linear projection from L. Then, π_L has positive dimensional general fiber.*

Proof. We do it only for the case $r = 2$. Let $q = \pi_L(p_1)$ and $L = T_{p_2}(X)$. By Terracini's Lemma, X is 2-defective if and only if $T_{p_1}(X) \cap T_{p_2}(X) \neq \emptyset$. Since $T_q(\pi_L(X)) = \pi_L(T_{p_1}(X))$, we deduce that $\dim T_q(\pi_L(X)) < \dim T_{p_1}(X)$. $\square$

Exercise 2.4. Prove Lemma 2.4.4 for $r > 2$.

Theorem 2.4.5. *Let $X \subseteq \mathbb{P}^n$ a r-defective projective variety. Let $\{p_1, \ldots, p_r\}$ be a general set of points on X. Then, $\Gamma_{p_1,\ldots,p_r}$ is positive dimensional.*

Proof. Let $L = \langle T_{p_2}(X), \ldots, T_{p_r}(X)\rangle$ and $q = \pi_L(p_1)$. By the Lemma 2.4.4, q has a fiber with a positive dimensional component passing through p_1. Since $\pi_L(\langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle) = T_q(\pi_L(X))$, then $\langle T_{p_1}(X), \ldots, T_{p_r}(X)\rangle$ is tangent along all fiber of q. $\square$

Therefore, we have that
$$r\text{-defective} \quad \Longrightarrow \quad r\text{-tangential defective} \quad \Longrightarrow \quad r\text{-weakly defective}.$$

Note that opposite directions does not hold. For example, a variety which is r-weakly defective but not r-defective is the degree-6 Veronese surface $\nu_6(\mathbb{P}^2)$. We will see it in Lecture III, see Remark 3.3.3.

Remark 2.4.6. The study of tangential contact loci is strictly related to the study of *identifiability*. A point $p \in \mathbb{P}^n$ is said to be **identifiable** with respect to an algebraic variety $X \subset \mathbb{P}^n$ if there exists a unique r-tuple $\{q_1, \ldots, q_r\} \subset X$ with $r = \mathrm{rk}_X(p)$ such that $p = \langle q_1, \ldots, q_r \rangle$. In [CO12, Proposition 2.4], it is proved that if X is not k-tangentially defective, i.e., $T_q(X) \subseteq \langle T_{p_1}(X), \ldots, T_{p_r}(X) \rangle$ if and only if $q = p_i$, then X is k-identifiable, i.e., the general element $p \in \sigma_k(X)$ is identifiable.

Lecture III

In Lecture II, we have seen that the problem of computing dimensions of secant varieties is translated to a linear algebra question on the intersection of general tangent spaces by Terracini's Lemma. With a further study of tangent spaces to Veronese varieties, the problem is translated to a *polynomial interpolation problem*.

3.1. **An interpolation problem: fat points.** Let consider the general framework of an integral polarized variety $(X, \mathcal{L})$ where $\mathcal{L}$ is a line bundle that defines an embedding of X into the (dual) projective space of its global sections. If $\{s_0, \ldots, s_d\}$ is a basis for the space of global sections $H^0(\mathbb{P}^n, \mathcal{L})$, then the embedding is

$$\varphi_{\mathcal{L}} : \mathbb{P}^n \longrightarrow \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{L})^\vee), \quad p \mapsto (s_0(p) : \ldots : s_d(p))$$

Example 3.1.1. Consider for example $(X, \mathcal{L}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ where the space of global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ corresponds to the space of degree-d homogeneous polynomials in $n+1$ variables. Hence, we consider the standard monomial basis of the space of global sections $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and the embedding $\varphi_{\mathcal{L}}$ corresponds to the Veronese embedding defined in Example 1.1.8. Similarly, the Segre-Veronese embeddings defined in Example 1.1.10 correspond to $(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}, \mathcal{O}_{\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}}(d_1, \ldots, d_m))$.

By definition, the pull-back of a hyperplane in $\mathbb{P}(H^0(X, \mathcal{L})^\vee)$ corresponds to an element of the linear system $\mathcal{L}$ on X. Let $p \in X$ and $q = \varphi_{\mathcal{L}}(p)$. If $H \subseteq \mathbb{P}(H^0(X, \mathcal{L})^\vee)$ is a hyperplane containing the tangent space $T_q(\varphi_{\mathcal{L}}(X))$, then the pull-back of H corresponds to an element $D = \varphi_L^*(H) \in \mathcal{L}$ which is *singular* at p. Indeed, if p is a smooth point, since $H \cap T_p(X) = T_p(X \cap H)$, then $X \cap H$ is singular at p if and only if $\dim T_p(X \cap H) > \dim(X \cap H) = \dim(X) - 1$, namely, $\dim H \cap T_p(X) \geq \dim X$, i.e., $T_p(X) \subseteq H$.

Example 3.1.2. Consider the point $p = (1 : 0 : \ldots : 0) \in \mathbb{P}^n$ defined by the ideal $I_p = (x_1, \ldots, x_n)$. Let $f = \sum_{|\alpha|=d} c_\alpha x^\alpha \in \Bbbk[x_0, \ldots, x_n]$ be a degree-d homogeneous polynomial in $n+1$ variables. Assume that f is singular at p, i.e., $\partial_{x_i}(f)(p) = 0$ for all $i \in \{0, \ldots, n\}$. By direct computation,

$$\partial_{x_0}(f)(p) = c_{d,0,\ldots,0} \quad \text{and} \quad \partial_{x_i}(f)(p) = c_{d-1,\ldots,1,\ldots,0}, \text{ for } i = 1, \ldots, n.$$

Therefore, f is singular at p if and only if $f \in (x_1, \ldots, x_n)^2$. I.e., f contains the 2-*fat point supported at* p.

Definition 3.1.3. Let X be a projective variety and $p \in X$. The m**-th fat point** supported at p, denoted mp, is the 0-dimensional scheme defined by the m-th power $\mathcal{I}_{p,X}^m$ of the ideal sheaf of $p \in X$.

Lemma 3.1.4. *Let $(X, \mathcal{L})$ be as above and such that $\mathcal{L}$ defines a closed embedding of X in $\mathbb{P}(H^0(X, \mathcal{L})^\vee)$. Let $N + 1 = \dim H^0(X, \mathcal{L})^\vee$. Then,*

$$\dim \sigma_s(\varphi_{\mathcal{L}}(X)) = N - \dim H^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$$

where $Z = 2p_1 + \ldots + 2p_s$ is a scheme of 2-fat points with general support.

Proof. By Terracini's Lemma, $\dim \sigma_s(\varphi_{\mathcal{L}}(X)) = \dim \langle T_{q_1}(\varphi_{\mathcal{L}}(X)), \ldots, T_{q_s}(\varphi_{\mathcal{L}}(X)) \rangle$. The latter corresponds to the codimension of the linear space of hyperplanes in $\mathbb{P}(H^0(X, \mathcal{L})^\vee)$ containing all the tangent spaces $T_{q_i}(\varphi_{\mathcal{L}}(X))$ for $i = 1, \ldots, s$. I.e.,

$$\dim \sigma_s(\varphi_{\mathcal{L}}(X)) = N - \dim \left\{ H \subseteq \mathbb{P}(H^0(X, \mathcal{L})^\vee) \ : \ H \text{ hyperplane}, \ H \supset T_{q_i}(X), \text{ for all } i = 1, \ldots, s \right\}.$$

As mentioned above, the condition $H \supset T_{q_i}(X)$ is equivalent to say that the pull-back of H is a divisor in $\mathcal{L}$ which is singular at $p_i = \varphi_{\mathcal{L}}^{-1}(q_i)$, i.e., $D = \varphi_{\mathcal{L}}^*(H) \in \mathcal{I}_{2p_i,X}$. □

3.2. **Defective Veronese varieties.** Following the previous notation, Veronese varieties are given by $(X, \mathcal{L}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Given any 0-dimensional scheme Z, we denote by $\mathcal{L}_{n,d}(Z)$ the linear system of degree-d hypersurfaces in $\mathbb{P}^n$ containing Z. In particular, if Z is a scheme of s 2-fat points with general support, then we write $\mathcal{L}_{n,d}(2^s) := \mathcal{L}_{n,d}(Z)$ for the *linear system of degree-d hypersurfaces of $\mathbb{P}^n$ that are singular at s general points.* For any linear system $\mathcal{L}$, we denote the *dimension of* $\mathcal{L}$ as $h^0(\mathcal{L}) := \dim H^0(\mathcal{L})$. Hence, by Lemma 3.1.4,

$$\dim \sigma_s(\nu_d(\mathbb{P}^n)) = N - h^0(\mathcal{L}_{n,d}(2^s)), \quad \text{with } N = \binom{n+d}{n} - 1.$$

Every singular point imposes $n+1$ conditions on the linear space of degree-d hypersurfaces of $\mathbb{P}^n$: the passage through the point and the vanishing of all tangent directions. Algebraically, these are given by the vanishing of the evaluations of the $n+1$ partial derivatives at the support points. Assuming that the p_i's are in general position, we may expect that all conditions are independent, i.e.,

$$\exp . h^0(\mathcal{L}_{n,d}(2^s)) = \max\{0, N+1-(n+1)s\}$$

which corresponds to the expected *co*dimension of $\sigma_s(\nu_d(\mathbb{P}^n))$ by Lemma 3.1.4.

In Example 2.3.2, we have seen a first example of defective variety by using directly Terracini's Lemma. We present here all defective cases Veronese varieties, explained from the polynomial interpolation interpretation.

Example 3.2.1 (Quadrics)**.** Consider $\mathcal{L}_{n,2}(2^s)$. Let $\{p_1,\ldots,p_s\}$ be a general set of points. Assume $s \leq n$. By Bèzout's Theorem, every hypersurface of $\mathcal{L}_{n,2}(2^s)$ has to be singular along all the lines $\langle p_i, p_j\rangle$ for $i \neq j$. Consequently, it has to be singular along all the linear span $\langle p_1,\ldots,p_s\rangle$. In other words, $\mathcal{L}_{n,2}(2^s)$ consits of quadratic cones with vertex $\langle p_1,\ldots,p_s\rangle$. That corresponds to the space of quadrics in $\mathbb{P}^{n-s}$. Hence, for $s \leq n$,

$$h^0(\mathcal{L}_{n,2}(2^s)) = h^0(\mathcal{L}_{n-s,2}) = \binom{n-s+2}{2}.$$

Therefore, we have that for $s \leq n$,

$$\begin{aligned}\dim \sigma_s(\nu_2(\mathbb{P}^n)) &= \binom{n+2}{2} - 1 - h^0(\mathcal{L}_{n,2}(2^s)) \\ &= \binom{n+2}{2} - 1 - \binom{n-s+2}{2} = s(n+1) - 1 - \binom{s}{2}\end{aligned}$$

i.e., $\nu_2(\mathbb{P}^n)$ s-defective with defect $\binom{s}{2}$ for any n.

Example 3.2.2 (Quartics in $\mathbb{P}^n$ for $n = 2,3,4$)**.** Consider $\mathcal{L}_{n,4}(2^s)$ with $s = \binom{n+2}{2} - 1$. For $n = 2,3,4$, we expect the linear system to be empty since, for $2 \leq n \leq 4$, we have

$$\binom{n+4}{4} - \left(\binom{n+2}{2} - 1\right) n \leq 0$$

equivalently, we expect $\sigma_s(\nu_4(\mathbb{P}^n))$ to fill the ambient space for $n = 2,3,4$. However, $\mathcal{L}_{n,2}(1^s)$ is 1-dimensional, indeed, by simple linear algebra, there is a unique quadric passing through $\binom{n+2}{2} - 1$ simple points in general position. Let D be such quadric. Then, $2D \in \mathcal{L}_{n,4}(2^s)$. It is easy to show computationally that this is indeed the unique quartic for a specific choice of the points and then, by semicontinuity, for general points. Therefore, $\sigma_s(\nu_4(\mathbb{P}^n))$ is s-defective with s-defect equal to 1 for $n = 2,3,4$.

Example 3.2.3 (Cubics in $\mathbb{P}^4$ with 7 general singularities)**.** Consider $\mathcal{L}_{4,3}(2^7)$. The linear system is again expected to be empty by parameter count, indeed, $\binom{4+3}{3} - 7\cdot 5 = 35 - 35 = 0$. However, it is classically known that there exists a unique Rational Normal Curve $\mathcal{C}_d \subseteq \mathbb{P}^d$ passing through $d+3$ general points, see [Har, Theorem 1.18]; in particular, there exists a $\mathcal{C}_4 \subseteq \mathbb{P}^4$ through seven general points. Up to a change of coordinates, the Rational Normal Curve is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix},$$

see [Har, Example 1.16]. Now, the 2-nd secant variety of $\mathcal{C}_4$ is defined by the determinant of the latter matrix, i.e., it is a cubic hypersurface which is singular along all the curve $\mathcal{C}_4$ and, in particular, at all the seven points. I.e., $\sigma_2(\mathcal{C}_4) \in \mathcal{L}_{4,3}(2^7)$. Again, this is the unique cubic hypersuface with seven singular points. Therefore, $\sigma_7(\nu_3(\mathbb{P}^4))$ is 7-defective with 7-defect equal to 1.

These defective cases were classically known. See for example the works by Campbell [Cam91], Palatini [Pal09], Terracini [Ter15], Castelnuovo [Cas89, Cas91] and Segre [Seg61]. However, the complete proof that they are the only defective Veronese varieties appeared only in 1995 after a series of papers by Alexander and Hirschowitz [Ale88, AH92b, AH92a, AH95].

Theorem 3.2.4 (Alexander-Hirschowitz Theorem, [AH95])**.** *The linear system $\mathcal{L}_{n,d}(2^s)$ has dimension as expected except in the following cases:*

- $d = 2$, *any* $n \geq 2$ *and* $2 \leq s \leq n$;
- $d = 3, n = 4, s = 7$;
- $d = 4, (n,s) \in \{(2,5),(3,9),(4,14)\}$.

By Lemma 3.1.4, the result is read as follows.

Corollary 3.2.5. *Let $d,n \in \mathbb{N}$. Then, the Waring rank of a general degree-d form in $n+1$ variables is*

$$\mathrm{wrk}^{\circ}(d,n) = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$$

except in the following cases:

- $\mathrm{wrk}^{\circ}(2,n+1) = n+1$*;*
- $\mathrm{wrk}^{\circ}(3,5) = 8$ *instead of* 7*;*
- $\mathrm{wrk}^{\circ}(4,n+1) = \binom{n+2}{2}$ *instead of* $\binom{n+2}{2}-1$ *for* $n = 2,3,4$*.*

3.3. **Plane curves.** In order to get familiar with the approach to defectivity of secant varieties through the study of linear systems with multiple base points, we consider here the case $n = 2$.

First, let us recall some easy, but important observations.

Remark 3.3.1. Let Z, Z', Z'' be 0-dimensional schemes such that $Z' \subseteq Z \subseteq Z''$.

(1) If Z imposes independent conditions on the linear system $\mathcal{L}$, i.e., $h^0(\mathcal{L}(Z)) = h^0(\mathcal{L}) - \deg(Z)$, then the same is true for Z'. Indeed, if, by contradiction $h^0(\mathcal{L}(Z')) > h^0(\mathcal{L}) - \deg(Z')$, then
$$\begin{aligned} h^0(\mathcal{L}(Z)) &\geq h^0(\mathcal{L}(Z')) - \deg(Z \smallsetminus Z') \\ &> h^0(\mathcal{L}) - \deg(Z') - \deg(Z \smallsetminus Z') = h^0(\mathcal{L}) - \deg(Z). \end{aligned}$$
(2) If $\mathcal{L}(Z)$ is empty, then $\mathcal{L}(Z'')$ is empty. Indeed, $\mathcal{L}(Z'') \subseteq \mathcal{L}(Z)$.

We prove Alexander-Hirschowitz Theorem in the plane.

Theorem 3.3.2. *The linear system $\mathcal{L}_{2,d}(2^s)$ is non-defective except for $(d,s) \in \{(2,2),(4,5)\}$. In particular, the general Waring rank for ternary forms of degree-d is*

$$\mathrm{wrk}^{\circ}(d,3) = \left\lceil \frac{\binom{d+2}{2}}{3} \right\rceil$$

except for:

- $d = 2$*, where* $\mathrm{wrk}^{\circ}(2,3) = 3$*, instead of* 2*;*
- $d = 4$*, where* $\mathrm{wrk}^{\circ}(4,3) = 6$*, instead of* 5*.*

Proof. The low degree cases can be approached directly.

The case $d = 1$ is trivial. The case $d = 2$ is Example 3.2.1.

Case $d = 3$. By Bèzout's Theorem, every cubic with two singular points contains all the line passing through them. It is immediate to observe that there is a unique cubic with three singularities: that is the union of the three lines passing through all pairs of the three base points. Hence, $h^0(\mathcal{L}_{2,3}(2^3)) = 1$ as expected. Hence, by Remark 3.3.1 we can immediately conclude that $\mathcal{L}_{2,3}(2^s)$ is non-defective for every s.

Case $d = 4$. We have seen in Example 3.2.1 that $h^0(\mathcal{L}_{2,4}(2^5)) = 1$ instead of 0: indeed, there is a unique quartic with five general singularities that is the the double conic passing through the five base points. Therefore, we immediately get $h^0(\mathcal{L}_{2,4}(2^s)) = 0$ for $s > 5$. It is easy to show that $h^0(\mathcal{L}_{2,4}(2^4)) = 3$ by finding three linearly independent totally reducible quartics (products of four lines) that do not have four general singular points: if $Z = 2p_1 + 2p_2 + 2p_3 + 2p_4$ are the four general 2-fat points and $\ell_{ij} = \langle p_i, p_j \rangle$, then we consider the four quartics $Q_i = \ell_{jh}\ell_{jk}\ell_{hk}\ell'_i \notin \mathcal{L}_{2,4}(Z)$, for $i = 1,\ldots,4$, where $\{j,h,k\} = \{1,\ldots,4\} \smallsetminus \{i\}$ and ℓ_i is a general line passing through p_i. By Remark 3.3.1, $\mathcal{L}_{2,4}(2^s)$ is non-defective for $s \leq 4$.

Case $d \geq 5$. We proceed by contradiction. By Remark 3.3.1, it is enough to consider the case $s = \left\lfloor \frac{\binom{2+d}{2}}{3} \right\rfloor$. Let $D \in \mathcal{L}_{2,d}(2^s)$. If $\mathcal{L}_{2,d}(2^s)$ was defective, by Lemma 2.4.2, we would have that D contains a double curve $2C$ where $C \in \mathcal{L}_{2,e}(1^s)$. Then, $2e \leq d$ and, since general simple points always impose independent conditions, $s \leq \binom{2+e}{2} - 1$. Hence,

$$\begin{aligned} \left\lfloor \frac{(d+2)(d+1)}{6} \right\rfloor = s &\leq \frac{(e+2)(e+1)}{2} - 1 \\ &\leq \frac{(\frac{d}{2}+2)(\frac{d}{2}+1)}{2} - 1 = \frac{d}{4}\left(\frac{d}{2}+3\right). \end{aligned}$$

The inequality $\left\lfloor \frac{(d+2)(d+1)}{6} \right\rfloor \leq \frac{d}{4}\left(\frac{d}{2}+3\right)$ holds only for $d \leq 4$ (already considered) and $d = 6$. For $d = 6$, the latter chain of inequalities forces $s = 9$. In this case, we know that there exists a unique cubic C passing through nine points and $2C$ is the unique sextic with nine general singularities. Hence, $h^0(\mathcal{L}_{2,6}(2^9)) = 1$ as expected. □

Remark 3.3.3 (Weakly defective variety, but non-defective)**.** The latter proof gives us also an example of a variety which is r-weakly defective but non-defective, see Section 2.4. Indeed, we have seen that a planar sextic singular at 9 general points is actually singular along the whole unic cubic passing through the points. In other words, fixing nine general points on $\nu_6(\mathbb{P}^2)$ we obtain a positive dimensional contact locus. At the same time, it is not defective because the expected dimension of $\mathcal{L}_{2,6}(2^9)$ is $\binom{2+6}{2} - 3 \cdot 9 = 28 - 27 = 1$.

3.4. **Cremona reductions.** After having considered binary forms (Remark 2.2.3) and ternary forms (Theorem 3.3.2), we move to quaternary forms. Since quadrics are defective (Example 3.2.1), we start by cubics.

Example 3.4.1 (Quaternary cubics)**.** Consider the linear system $\mathcal{L}_{3,3}(2^s)$. By Remark 3.3.1, it is enough to consider the case $s = 5$ for which we expect the linear system to be empty. For $s = 4$, by semicontinuity, we may consider the scheme of 2-fat points supported at the four coordinate points in $\mathbb{P}^3$ and it is easy to prove that $h^0(\mathcal{L}_{3,3}(2^4)) = 4$ by showing that a base for the linear system is given by the cubics $\langle x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3\rangle$. Now, consider the Cremona transformation

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3, \quad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0x_1x_2 : x_0x_1x_3 : x_0x_2x_3 : x_1x_2x_3). \tag{3.9}$$

Cubic surfaces singular at the four coordinate points are mapped to hyperplanes. If we assume the existence of a cubic surface with a fith singular point, i.e., if we assume that $h^0(\mathcal{L}_{3,3}(2^5)) > 0$, then such cubic would be mapped to a hyperplane with a singular point, which is clearly absurd.

Remark 3.4.2 (Cremona Reduction)**.** The latter idea of using Cremona transformations to transform a linear system to another with the same dimension and, hopefully, easier to compute is general. Generalizing (3.9), in $\mathbb{P}^n$, we consider the standard Cremona transformation

$$Cr_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \quad (x_0 : \ldots : x_n) \mapsto (x_0^{-1} : \ldots : x_n^{-1}) = \left(\frac{x_0 \cdots x_n}{x_0} : \ldots : \frac{x_0 \cdots x_n}{x_n}\right).$$

Given a scheme of general fat points $Z = m_0p_0 + \ldots + m_sp_s$ with $s \geq n+1$, we write $\mathcal{L}_{n,d}(m_0, \ldots, m_s) := \mathcal{L}_{n,d}(Z)$. Without loss of generality, we may assume that the first $n+1$ of the base points are the coordinate points. Then, the following useful lemma holds. See also [LU06].

Lemma 3.4.3 (Cremona Reduction Lemma)**.** *Fixed integers $n, d, m_0, \ldots, m_s$, let $k = (n-1)d - m_0 - \ldots - m_s$. Then,*

$$Cr_n(\mathcal{L}_{n,d}(m_0, \ldots, m_s)) = \mathcal{L}_{n,d+k}(m_0 + k, \ldots, m_n + k, m_{n+1}, \ldots, m_s)$$

Proof. Since the Cremona transformation is an isomorphism outside the base locus, it is enough to prove the case $s = n$. This can be shown by observing that if $\{x_0, \ldots, x_n\}$ are the cooridinates on $\mathbb{P}^n$, then, for any $(a_0, \ldots, a_n) \in \mathbb{N}^{n+1}$ with $\sum_i a_i = d$, we have

$$Cr_n(x_0^{a_0} \cdots x_n^{a_n}) = x_0^{d-a_0} \cdots x_n^{d-a_n}.$$

Now, a basis of the linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_{n+1})$ is given by all degree-d monomials $x_0^{a_0} \cdots x_n^{a_n}$ such that $0 \leq a_i \leq d - m_i$. Under such constraints, we can additionally write

$$Cr_n(x_0^{a_0} \cdots x_n^{a_n}) = (x_0^{m_0} \cdots x_n^{m_n}) \cdot (x_0^{d-a_0-m_0} \cdots x_n^{d-a_n-m_n}).$$

Note that, $\sum_i (d - a_i - m_i) = d - k$. Moreover, since $0 \leq a_i \leq d - m_i$, we have

$$0 \leq d - a_i - m_i \leq (d+k) - (m_i + k),$$

i.e., the monomial $x_0^{d-a_0-m_0} \cdots x_n^{d-a_n-m_n}$ is an element of the monomial basis of $\mathcal{L}_{n,d+k}(m_0 + k, \ldots, m_n + k)$. In other words, the Cremona map induces a bijection between the two monomial basis. □

In Example 3.4.1, we considered $\mathcal{L}_{3,3}(2^5)$. Then, Lemma 3.4.3 tells us exactly that

$$Cr_3(\mathcal{L}_{3,3}(2^4)) = \mathcal{L}_{3,1}(2)$$

because $k = 2 \cdot 3 - 2 \cdot 4 = -2$.

Therefore, whenever we have a linear system $\mathcal{L}_{n,d}(m_0, \ldots, m_s)$ for which $k = (n-1)d - m_0 - \ldots - m_s < 0$, then we may transform the linear system to an equidimensional linear system on hypersurfaces with lower degree and with simpler singularities. On the other hand, if $k \geq 0$, we say that $\mathcal{L}_{n,d}(m_0, \ldots, m_s)$ is Cremona reduced.

Lecture IV

In Lecture III, we translated the study of defective Veronese varieties to polynomial interpolation problems. We presented the list of all defective cases and we proved Alexander-Hirschowitz Theorem in the case of planar curves. In higher dimensional spaces, we need more tools in order to prove inductively results of non-defectiveness.

4.1. **Degeneration techniques for non-defectiveness.**

In order to prove that a linear system with multiple base points has dimension as expected we can argue by degeneration.

Let $\mathcal{L}$ be a linear system of divisors in $\mathbb{P}^n$ and, for any scheme $Z \subseteq \mathbb{P}^n$, let $\mathcal{L}(Z)$ be the linear subsystem of divisors in $\mathcal{L}$ that contain Z. If $\tilde{Z}$ is a degeneration of Z, by semicontinuity, the dimension can only increase, i.e.,

$$\exp.\dim \mathcal{L}(Z) \leq \dim \mathcal{L}(Z) \leq \dim \mathcal{L}(\tilde{Z}). \tag{4.10}$$

Hence, in order to prove that $\mathcal{L}(Z)$ has dimension as expected, we may look for a specialization $\tilde{Z}$ for which we can prove that left and right-hand-side of (4.10) coincide.

4.1.1. *Castelnuovo's exact sequence and "la mèthode d'Horace".* The first idea, which goes back to Castelnuovo [Cas91], is to specialize the support of (some of) the base points of the linear system to lie on a hyperplane or, more generally, on a subvariety. Given a subvariety $H \subseteq \mathbb{P}^n$, for any 0-dimensional scheme Z, we have the following exact sequence of line bundles, known as *Castelnuovo's exact sequence*,

$$0 \to \mathcal{L}(Z,H) \to \mathcal{L}(Z) \to \mathcal{L}(Z)|_H \to 0$$

where $\mathcal{L}(Z)|_H$ is the restriction on H and $\mathcal{L}(Z,H)$ is the linear subsystem of divisors of $\mathcal{L}(Z)$ containing H. By passing to homology, we obtain

$$0 \to H^0(\mathcal{L}(Z,H)) \to H^0(\mathcal{L}(Z)) \to H^0(\mathcal{L}(Z)|_H)$$

and, therefore, we get

$$h^0(\mathcal{L}(Z)) \leq h^0(\mathcal{L}(Z,H)) + h^0(\mathcal{L}(Z)|_H). \tag{4.11}$$

Let us go back to our case. Let $Z \subseteq \mathbb{P}^n$ be any 0-dimensional scheme and let $\mathcal{L} = \mathcal{L}_{n,d}$ be the linear system of degree-d hypersurfaces in $\mathbb{P}^n$. Assume $H \cong \mathbb{P}^{n-1}$ to be a hyperplane. We call:

- **residue** of Z with respect to H the scheme obtained by *removing* from Z its intersection with H, i.e., the scheme whose defining ideal is $I(Z) : H$ and denoted $\mathrm{Res}_H(Z)$;
- **trace** of Z with respect to H the schematic intersection $Z \cap H$ and denoted $\mathrm{Tr}_H(Z)$.

Therefore, it is immediate to see that:

- $\mathcal{L}_{n,d}(Z,H) \cong \mathcal{L}_{n,d-1}(\mathrm{Res}_H(Z))$;
- $\mathcal{L}_{n,d}(Z)|_H \cong \mathcal{L}_{n-1,d}(\mathrm{Tr}_H(Z))$.

Remark 4.1.1. If $Z = mp$ is a m-fat point in $\mathbb{P}^n$ and $p \in H$ where H is a hyperplane, then:

- $\mathrm{Res}_H(Z)$ is a $(m-1)$-fat point in $\mathbb{P}^n$;
- $\mathrm{Tr}_H(Z)$ is a m-fat point in $\mathbb{P}^{n-1}$.

Assume, $p = (1:0:\ldots:0)$, $Z = mp$ defined by $I(Z) = (x_1,\ldots,x_n)^m$ and $H = \{x_n = 0\}$. Then:

- $\mathrm{Res}_H(Z)$ is defined by the ideal $(x_1,\ldots,x_n)^m : (x_n) = (x_1,\ldots,x_n)^{(m-1)}$;
- $\mathrm{Tr}_H(Z)$ is defined by $(x_1,\ldots,x_n)^m \otimes \Bbbk[x_0,\ldots,x_n]/(x_n) = (x_1,\ldots,x_{n-1})^m \subseteq \Bbbk[x_0,\ldots,x_{n-1}]$.

Therefore, combining (4.10) and (4.11), we obtain the following:

$$\exp.\dim \mathcal{L}_{n,d}(Z) \leq \dim \mathcal{L}_{n,d}(Z) \leq \dim \mathcal{L}_{n,d-1}(\mathrm{Res}_H(\tilde{Z})) + \dim \mathcal{L}_{n-1,d}(\mathrm{Tr}_H(\tilde{Z})). \tag{4.12}$$

Note that on the right-hand-side we have linear systems of hypersurfaces either of lower degree or in lower dimension. Therefore, (4.12) is the core of an approach by double induction on (d,n) where the specialization $\tilde{Z}$ is carefully chosen. Let us see it in a concrete example.

Example 4.1.2 (Quaternary quartics)**.** Let $n = 3, d = 4$. We have seen in Example 3.2.2 that $\mathcal{L}_{3,4}(2^9)$ is defective and 1-dimensional, i.e., $\sigma_9(\nu_4(\mathbb{P}^3))$ is a hypersurface instead of filling the ambient space as expected. From this, it is trivial that $\mathcal{L}_{3,4}(2^s)$ is empty, as expected, for $s \geq 9$. Consider $s = 8$. The expected dimension is

$$\exp.h^0\left(\mathcal{L}_{3,4}(2^8)\right) = \max\left\{0, \binom{3+4}{3} - 8(3+1)\right\} = \max\{0, 35-32\} = 3.$$

Let $H \cong \mathbb{P}^2$ be a general plane. Let Z be a scheme of eight 2-fat points and let $\tilde{Z}$ be a specialization of Z such that $\tilde{Z} = 2p_1 + \ldots + 2p_8$ with $p_1, p_2, p_3, p_4 \in H$ and p_5, p_6, p_7, p_8 general points outside H. Therefore:

$$\mathrm{Res}_H(\tilde{Z}) = p_1 + \ldots + p_4 + 2p_5 + \ldots + 2p_8 \subseteq \mathbb{P}^3 \quad \text{and} \quad \mathrm{Tr}_H(\tilde{Z}) = 2p_5 + \ldots + 2p_8 \subseteq H.$$

By Theorem 3.3.2, we know that $h^0(\mathcal{L}_{2,4}(\mathrm{Tr}_H(Z)) = h^0(\mathcal{L}_{2,4}(2^4)) = \binom{2+4}{4} - 4\cdot 3 = 15 - 12 = 3$.

Now, we look at $\mathcal{L}_{3,3}(\mathrm{Res}_H(\tilde{Z}))$. The linear system $\mathcal{L}_{3,3}(2^4) = \mathcal{L}_{3,3}(2p_5+\ldots+2p_8)$ is non-defective. In particular, $h^0(\mathcal{L}_{3,3}(2^4)) = 4$. The four extra simple points $p_1, \ldots, p_4$ on H impose four independent conditions on $\mathcal{L}_{3,3}(2^4)$. Indeed, a general point on H fails to impose a condition on a linear system if all divisors in the linear system contain already H. In our case, there are no elements of $\mathcal{L}_{3,3}(2^4)$ that are union of a quadric and H since, by Example 3.2.1, $h^0(\mathcal{L}_{3,2}(2^4)) = 0$. Hence,

$$3 = \exp.h^0\left(\mathcal{L}_{3,4}(2^8)\right) \leq h^0(\mathcal{L}_{3,3}(\mathrm{Res}_H(Z))) + h^0(\mathcal{L}_{2,4}(\mathrm{Tr}_H(Z))) = 0 + 3.$$

By Remark 3.3.1, we conclude that $\mathcal{L}_{3,4}(2^s)$ is non-defective also for $s \leq 8$.

Remark 4.1.3 (La mèthode d'Horace)**.** The degeneration approach followed in Example 4.1.2 was introduced by Hirschowitz [Hir85] under the name of *Horace method*. An ancient legend from the Roman kingdom era narrates of the war between Rome and the neighboring city of Alba Longa. In order to avoid a bloody war between their armies, it was decided that the war would have been decided by an epic clash between the two triplets of Roman Horatii and the Alban Curiatii. Shortly after the beginning of the fight, only one of the Horatii survived while the Curiatii were left wounded in different ways. Publius' idea was to start running so that, during the chase, the Curiatii would have been splitted a part due to their different speeds. This allowed Publius' to fight the Curiatii one by one and win the fight. Hirschowitz probably felt that he could not deal with all the required singularities by one and, inspired by the legend, he started to split them in smaller schemes and approach them one by one.

As already mentioned, the Horace method and the use of Castelnuovo's Exact Sequence allows us to follow a double-induction approach on degree and dimension. However, as we have noticed in Example 4.1.2, the residual scheme is in special position: it is composed by a union of general 2-fat points and a set of simple points which are general on a hyperplane. Similarly as in Example 4.1.2, the conditions imposed by the union of general 2-fat points is deduced by induction, but we need a way to deal with the simple points specialized on a hyperplane.

Lemma 4.1.4 ([CGG07, Lemma 1.9])**.** *Let $Z \subseteq \mathbb{P}^n$ be a 0-dimensional scheme and let H be a hyperplane. Let $p_1, \ldots, p_s \in H$ be general points on H. Let $Z' = \mathrm{Res}_H(Z)$.*

1. *If $h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_{s-1})) > h^0(\mathcal{L}_{d-1}(Z'))$, then $h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_s)) = h^0(\mathcal{L}_d(Z)) - s$.*
2. *If $h^0(\mathcal{L}_d(Z)) \leq s$ and $h^0(\mathcal{L}_{d-1}(Z')) = 0$, then $h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_s)) = 0$.*

Proof. (1) We proceed by induction on s. If $s = 1$ and $h^0(\mathcal{L}_d(Z)) > h^0(\mathcal{L}_{d-1}(Z'))$ then there is a hypersurface in $\mathcal{L}_d(Z)$ which does not contain H. Hence, p_1 imposes one condition and $h^0(\mathcal{L}_d(Z+p_1)) = h^0(\mathcal{L}_d(Z)) - 1$. Assume $s > 1$. Since $h^0(\mathcal{L}_d(Z+p_1+\ldots+p_{s-1})) > h^0(\mathcal{L}_{d-1}(Z'))$, there is a hypersurface in $\mathcal{L}_d(Z+p_1+\ldots+p_{s-1})$ which does not contain H. Hence, $h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_s)) = h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_{s-1})) - 1$. Now, by induction,

$$h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_s)) = h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_{s-1})) - 1 = h^0(\mathcal{L}_d(Z)) - s.$$

(2) If $h^0(\mathcal{L}_d(Z)) = 0$ then it is obvious. Assume $h^0(\mathcal{L}_d(Z)) = v > 0$. Hence, $h^0(\mathcal{L}_d(Z + p_1 + \ldots + p_{v-1})) > 0 = h^0(\mathcal{L}_{d-1}(Z'))$. By (1), $h^0\mathcal{L}_d(Z + p_1 + \ldots + p_v) = h^0(\mathcal{L}_d(Z)) - v = 0$. Since $s \geq v$, the claim follows. □

Exercise 4.5 (Quaternary quintics)**.** Show that $\mathcal{L}_{3,5}(2^s)$ is never defective.

4.1.2. *La mèthode d'Horace differentielle.* After having considered the case of quaternary quartics and quintics, we consider the case of sextics. We notice very quickly that the Horace method described in the previous section cannot succeed due to numerical constraints.

Example 4.1.5 (Quaternary sextic: failure of Horace method)**.** Since $h^0(\mathcal{L}_{3,6}) = \binom{3+6}{3} = 84$. Since a 2-fat point imposes four conditions, by Remark 3.3.1, it is enough to consider the case $\mathcal{L}_{3,6}(2^{21})$ which is expected to be empty. Again, let $Z = 2p_1 + \ldots + 2p_t + 2p_{t+1} + \ldots + 2p_s$ with $\{p_1, \ldots, p_t\} \subseteq H$ general points on a general hyperplane H and $\{p_{t+1}, \ldots, p_s\}$ general points outside H. Then, the inequality (4.12) becomes

$$0 \leq h^0(\mathcal{L}_{3,6}(2^{21})) \leq h^0(\mathcal{L}_{3,5}(\mathrm{Res}_H(Z))) + h^0(\mathcal{L}_{2,6}(\mathrm{Tr}_H(Z))). \tag{4.13}$$

By Theorem 3.3.2, $h^0(\mathcal{L}_{2,6}(\mathrm{Tr}_H(Z))) = \max\{0, 28 - 3t\}$ which is equal to 0 if and only if $t \geq 10$. On the other hand, if $t \geq 10$, then

$$\begin{aligned} h^0(\mathcal{L}_{3,5}(\mathrm{Res}_H(Z))) &\geq 56 - \deg(\mathrm{Res}_H(Z)) \\ &= 56 - 4(21 - t) - t = -28 + 3t \geq 2. \end{aligned}$$

In other words, there is no way for the right-hand-side of (4.13) to be equal to 0.

Within a series of enlightening papers [Ale88, AH92b, AH92a, AH95, AH00], Alexander and Hirschowitz developed a stronger version of the Horace method which allows to overcome the arithmetic issues observed in Example 4.1.5. Roughly speaking, the idea is to degenerate a 2-fat point on a hyperplane H in such a way the trace on H is not a 2-fat point on H, but it is a simple point. We follow [AH00, Section 9].

Let $Z = mp$ be a m-fat point supported at $(1 : 0 : \ldots : 0) \in \mathbb{P}^n$. The ideal $I = I(Z) = (x_1, \ldots, x_n)^m$ can be decomposed as

$$I = (x_1, \ldots, x_{n-1})^m \oplus (x_1, \ldots, x_{n-1})^{m-1} \cdot x_n \oplus \ldots \oplus (x_1, \ldots, x_{n-1}) \cdot x_n^{m-1} \oplus (x_n^m).$$

This trivial algebraic description is translated geometrically by saying that Z is *vertically graded* with respect to x_n (and actually to any $x_1, \ldots, x_n$) in the sense that Z can be constructed by infinitesimally piling up a series of j-th fat points on $\{x_n = 0\}$ for $j \in \{0, \ldots, m-1\}$. We say that these are the *slices* of Z. Only the first layer, the one corresponding to $(x_1, \ldots, x_{n-1})^m$ belongs to H and that is indeed the trace of Z on H as interpreted in Remark 4.1.1. The heuristic idea of the *differential Horace method* is that Z can actually be degenerated in such a way the trace on H can be any of its slices. In other words, for any $p \geq 0$, we define:

- the **p-th residue** of Z with respect to H as the 0-dimensional scheme defined by
$$I(\mathrm{Res}^p_H(Z)) = I(Z) + (I(Z) : I(H)^{p+1})I(H)^p;$$
- the **p-th trace** of Z on H as the 0-dimensional scheme defined by
$$I(\mathrm{Tr}^p_H(Z)) = (I(Z) : I(H)^p) \otimes \Bbbk[x_0, \ldots, x_n]/I(H).$$

For $p = 0$, these are the residue and trace defined in Remark 4.1.1.

Let us see the case $p = 1$ in the case 2-fat points.

Example 4.1.6. Let $Z = 2p \subseteq \mathbb{P}^n$ with $p = (1 : 0 : \ldots : 0)$ defined by $(x_1, \ldots, x_n)^2$. Then:

- the 1-st residue is
$$\begin{aligned} I(\mathrm{Res}^1_H(Z)) &= (x_1, \ldots, x_n)^2 + ((x_1, \ldots, x_n)^2 : (x_n)^2)x_n \\ &= (x_1, \ldots, x_n)^2 + (x_n) = (x_1, \ldots, x_{n-1})^2 + (x_n), \end{aligned}$$
i.e., it is a 0-dimension scheme in $\mathbb{P}^n$ which is equal to a 2-fat point on H;
- the 1-st trace is
$$\begin{aligned} I(\mathrm{Tr}^1_H(Z)) &= \big((x_1, \ldots, x_n)^2 : (x_n)\big) \otimes \Bbbk[x_0, \ldots, x_n]/(x_n) \\ &= (x_1, \ldots, x_n) \otimes \Bbbk[x_0, \ldots, x_n]/(x_n), \end{aligned}$$
i.e., it is the simple point on H supported at the origin.

Remark 4.1.7. The latter idea can be generlized to any 0-dimensional scheme that is vertically graded with respect to a smooth divisor passing through its support.

The differential Horace Lemma for 2-fat points goes as follows.

Lemma 4.1.8 (Differential Horace Lemma, [AH00])**.** *Let $H \subseteq \mathbb{P}^n$ be a hyperplane. Let Z be a 0-dimensional scheme and let $p_1, \ldots, p_t$ be general points on H. Let $D_H(p_i) = 2p_i \cap H$. Assume that:*

(1) $h^0(\mathcal{L}_{n,d-1}(\mathrm{Res}_H(Z) + D_H(p_1) + \ldots + D_H(p_t))) = 0$,
(2) $h^0(\mathcal{L}_{n-1,d}(\mathrm{Tr}_H(Z) + p_1 + \ldots + p_t)) = 0$,

then, $h^0(\mathcal{L}_{n,d}(Z)) = 0$.

Example 4.1.9 (Quaternary sextics: via differential Horace Lemma)**.** Consider $\mathcal{L}_{3,6}(2^{21})$ which is expected to be empty. As we have seen in Example 4.1.5, the standard Horace method fails because airthmetically we were overshoothing two conditions by specializing 10 among the 21 base points. Therefore, consider a degeneration of the general scheme of 21 2-fat points by specializing 9 points in the classic way and one differentially. I.e., let H be a general hyperplane and let Z be a scheme of 21 2-fat points such that

$$\mathrm{Res}_H(Z) = p_1 + \ldots + p_9 + D_H(p_{10}) + 2p_{11} + \ldots + 2p_{21}$$

and

$$\mathrm{Tr}_H(Z) = 2p_1 + \ldots + 2p_9 + p_{10} \subseteq H.$$

- Consider $\mathcal{L}_{2,6}(\mathrm{Tr}_H(Z))$. By Theorem 3.3.2, we know that $h^0(\mathcal{L}_{2,6}(2^9)) = 28 - 9 \cdot 3 = 1$. Hence,
$$h^0(\mathcal{L}_{2,6}(\mathrm{Tr}_H(Z))) = 0.$$

- Consider $\mathcal{L}_{3,5}(\mathrm{Res}_H(Z))$. We apply Lemma 4.1.4(ii). Note that
$$D_H(p_{10}) + 2p_{11} + \ldots + 2p_{21} \subseteq 2p_{10} + 2p_{11} + \ldots + 2p_{21}$$
where $p_{10}, \ldots, p_{21}$ are general points in $\mathbb{P}^3$. Hence, by Exercise 4.5, $h^0(\mathcal{L}_{3,5}(2^{12})) = 56 - 12 \cdot 4 = 8$. By Remark 3.3.1, $h^0(\mathcal{L}_{3,5}(D_H(p_{10}) + 2p_{11} + \ldots + 2p_{21})) = 9$. Moreover, by Example 4.1.2 and Remark 3.3.1, $h^0(\mathcal{L}_{3,4}(p_{10} + 2p_{11} + \ldots + 2p_{21})) = 0$. Therefore, by Lemma 4.1.4(ii), we deduce
$$h^0(\mathcal{L}_{3,5}(\mathrm{Res}_H(Z))) = 0.$$

Hence, we conclude that $\mathcal{L}_{3,6}(2^{21})$ is empty by (4.12). By Remark 3.3.1, $\mathcal{L}_{3,6}(2^s)$ is never defective.

Remark 4.1.10. After the successfull approach introduced by Alexander and Hirschowitz, the same methodology has been applied also to other cases, such as the study of dimensions of secant varieties of Segre and Segre-Veronese varieties or to study dimensions of linear systems of hypersurfaces with base points of higher multiplicities. At the same time, different degeneration techniques have been proposed leading to different proofs of Alexander-Hirschowitz Theorem. For example, Chandler's *Curvilinear Lemma* [Cha01] or Brambilla and Ottaviani degeneration of supports over higher codimensional varieties [BO08]. The latter led to the classification of defective tangential varieties of Veronese varieties [AV18].Postinghel [Pos12] exploited a degeneration of the projective space to a reducible variety via a blow-up construction, see [Ran89, CM98, CM98]. Another degeneration technique, which can be explained also through a blow-up construction and a degeneration of the projective space to a reducible variety, was due to Evain [Eva97] who considered collisions of (part of) the base points of the linear system: this has been exploited recently to deduce a non-defectiveness criterion in [GO22].

Remark 4.1.11. It is worth to mention that the latter dgeneration technique mentioned in Remark 4.1.10 has been also used by Galuppi and Mella in [GM19]. In this work, the authors completed the classification of pairs (d, n) for which the general degree-d homogeneous polynomial in $n+1$ variables is identifiable, i.e., it admits a unique minimal Waring decomposition, see Remark 2.4.6. In particular, they show that the only special cases for which a general form is rank-r identifiable are:

- odd binary forms ($n = 1$, $d = 2s - 1$, $r = s$), see Proposition 5.2.7;
- quaternary cubics ($n = 3, d = 3, r = 5$), known as Sylvester's Pentehedron Theorem, see [Dol12, Section 9.4.1];
- ternary quintics ($n = 2, d = 5, r = 7$), see [Pal03].

4.2. **Segre varieties and multiprojective linear systems.** We conclude this lecture by commenting also on defectiveness of Segre and Segre-Veronese varieties. As recalled in Example 1.1.10, Segre-Veronese varieties deal with partially-symmetric tensors. This can be rephrased by considering multigraded polynomials.

Indeed, if $\mathbf{x}_i = (x_{i,0}, \ldots, x_{i,n_i})$ are tuples of variable, let
$$S = \Bbbk[\mathbf{x}_1; \ldots; \mathbf{x}_m] = \Bbbk[\mathbf{x}_1] \otimes \cdots \otimes \Bbbk[\mathbf{x}_m] = \bigoplus_{i_1,\ldots,i_m} S_{i_1,\ldots,i_m}$$
where $S_{i_1,\ldots,i_m}$ is the $\Bbbk$-vector space of multihomogeneous polynomials of multidegree $(i_1, \ldots, i_m)$.

A tensor $T = (t_{j_1,\ldots,j_m}) \in \Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_m}$ can be regarded as the multihomogeneous polynomial of multidegree $(1, \ldots, 1)$ given by $\sum_{j_1,\ldots,j_m} t_{j_1,\ldots,j_m} x_{1,j_1} \cdots x_{m,j_m}$. In particular, if $\mathbf{v}_i = (v_{i,0}, \ldots, v_{i,n_i}) \in \Bbbk^{n_i+1}$ are vectors, the rank-one tensor $T = \mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_m$ can be regarded as the multihomogeneous polynomial $\ell_1 \cdots \ell_m$ of multidegree $(1, \ldots, 1)$ where $\ell_i = v_{i,0}x_{i,0} + \ldots + v_{i,m}x_{i,m} \in \Bbbk[\mathbf{x}_i]$ are linear forms in the i-th set of variables.

Since we want to approach the computation of dimensions of secant varieties via Terracini's Lemma to study dimensions of secant varieties of Segre-Veronese varieties, we need first to understand tangent space to the Segre-Veronese varieties. To easy the notation, we write it in terms of multihomogeneous polynomials.

Let $\ell_1^{d_1} \cdots \ell_m^{d_m} \in \nu_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})$ where $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$. Then,
$$\left.\frac{d}{dt}\right|_{t=0} (\ell_1 + th_1)^{d_1} \cdots (\ell_m + th_m)^{d_m} = \ell_1^{d_1-1} h_1 \ell_2^{d_2} \cdots \ell_m^{d_m} + \ell_1^{d_1} \ell_2^{d_2} h_2 \cdots \ell_m^{d_m} + \ldots + \ell_1^{d_1} \cdots \ell_{m-1}^{d_{m-1}} \ell_m^{d_m-1} h_m$$
$$= \sum_{i=0}^{m} \frac{\ell_1^{d_1} \cdots \ell_m^{d_m}}{\ell_i} h_i$$
hence, if $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$ is the coordinate vector with 1 in i-th entry,
$$T_{[\ell_1^{d_1} \cdots \ell_m^{d_m}]} \nu_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}) = \left\langle S_{\mathbf{e}_i} \frac{\ell_1^{d_1} \cdots \ell_m^{d_m}}{\ell_i} \ : \ i = 0, \ldots, m \right\rangle$$

By Lemma 3.1.4, dimensions of secant varieties to Segre-Veronese varieties are related to the study of multigraded hypersurfaces with multiple base points. In other words, if $p_1, \ldots, p_s \in \mathbb{P}^{\mathbf{n}}$ are general points, then

$$\dim \sigma_s(\nu_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) = \prod_{i=1}^{m}(n_i+1) - h^0(\mathcal{L}_{\mathbf{n},\mathbf{d}}(2^s))$$

where $\mathcal{L}_{\mathbf{n},\mathbf{d}}(2^s)$ is the linear system of hypersurfaces in $\mathbb{P}^{\mathbf{n}}$ of multidegree $\mathbf{d}$ through s general 2-fat points.

Catalisano, Geramita and Gimigliano introduced the *multi-affine-projective method*, see [CGG05]. This method allows to associate to the multigraded linear system $\mathcal{L}_{\mathbf{n},\mathbf{d}}(2^s)$ a linear system in $\mathbb{P}^{n_1+\ldots+n_m}$. The idea is as follows. Let $N = \sum_i n_i$. We consider the birational map obtained by composing the birational map

$$\begin{array}{ccc} \mathbb{P}^{\mathbf{n}} & \dashrightarrow & \mathbb{A}^N, \\ ((x_{1,0}:\ldots:x_{1,n_1}),\ldots,(x_{m,0}:\ldots:x_{m,n_m})) & \mapsto & \left(\frac{x_{1,1}}{x_{1,0}},\ldots,\frac{x_{1,n_1}}{x_{1,0}},\ldots,\frac{x_{m,1}}{x_{m,0}},\ldots,\frac{x_{m,n_m}}{x_{m,0}}\right), \end{array}$$

with the natural embedding of $\mathbb{A}^N$ into the chart of $\mathbb{P}^N$ given by the first entry different from 0, i.e.,

$$\begin{array}{ll} \mathbb{A}^N & \left(\frac{x_{1,1}}{x_{1,0}},\ldots,\frac{x_{1,n_1}}{x_{1,0}},\ldots,\frac{x_{m,1}}{x_{m,0}},\ldots,\frac{x_{m,n_m}}{x_{m,0}}\right) \\ \downarrow & \qquad\qquad \mapsto \\ \mathbb{P}^N & \left(1:\frac{x_{1,1}}{x_{1,0}}:\ldots:\frac{x_{1,n_1}}{x_{1,0}}:\ldots:\frac{x_{m,1}}{x_{m,0}}:\ldots:\frac{x_{m,n_m}}{x_{m,0}}\right) = \\ & (x_{1,0}\cdots x_{m,0}:x_{1,1}\cdots x_{m,0}:\ldots:x_{1,n_1}\cdots x_{m,0}:\ldots:x_{1,0}\cdots x_{m,1}:\ldots:x_{1,0}\cdots x_{m,n_m}) \end{array}$$

Let's call such composition $\psi : \mathbb{P}^{\mathbf{n}} \dashrightarrow \mathbb{P}^N$.

If we label the coordinates of $\mathbb{P}^N$ by $\{z_0, z_{1,1}, \ldots, z_{1,n_1}, \ldots, z_{m,1}, \ldots, z_{m,n_m}\}$ and let $q_0, q_{1,1}, \ldots, q_{m,n_m}$ the corresponding coordinate points in $\mathbb{P}^N$. Then, for $i = 1, \ldots, m$, consider the linear spaces

$$\Pi_i = \langle q_{i,1}, \ldots, q_{i,n_i}\rangle \cong \mathbb{P}^{n_i-1}.$$

i.e., $I(\Pi_i) = (z_0, z_{1,1}, \ldots, \widehat{z_{i,1}}, \ldots, \widehat{z_{i,n_i}}, \ldots, z_{m,n_m})$.

Lemma 4.2.1 ([CGG05])**.** *Let $Z \subseteq \mathbb{P}^{\mathbf{n}}$ be a 0-dimensional scheme with support within the chart $x_{1,0}\cdots x_{m,0} \neq 0$ and $Z' = \psi(Z) \subseteq \mathbb{P}^N$. Let $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ and $D = \sum_i d_i$. Let*

$$W = Z' + (D-d_1)\Pi_1 + \ldots + (D-d_m)\Pi_m \subseteq \mathbb{P}^N.$$

Then,

$$h^0(\mathcal{L}_{\mathbf{n},\mathbf{d}}(Z)) = h^0(\mathcal{L}_{N,D}(W)).$$

The latter approach has the advantage of considering again linear systems in standard graded setting, as in the case of Veronese varieties, but, at the same time, it has a clear draw-back: even if we start from the multigraded Hilbert function of a 0-dimensional scheme, we might end up to a standard graded Hilbert function of higher-dimensional non-reduced spaces. The latter issue is avoided in the case of products of $\mathbb{P}^1$'s for which this approach allowed for a complete classification of defective cases.

4.3. **The case of $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$.** In the special case of $\mathbf{n} = (1, \ldots, 1)$, all linear spaces Π_i in Lemma 4.2.1 are 0-dimensional. Namely, combining Lemma 3.1.4 and Lemma 4.2.1, we get the following.

Corollary 4.3.1. *Let $\mathbf{d} = (d_1, \ldots, d_m)$ and $D = \sum_i d_i$. Let $Z = d_1q_1 + \ldots + d_mq_m + 2p_1 + \ldots + 2p_s \subseteq \mathbb{P}^m$ be a scheme of fat points where q_i is the i-th coordinate point and the p_i's are general. Then, the s-th secant variety to the Segre-Veronese variety $\nu_{\mathbf{d}}(\mathbb{P}^{\mathbf{1}})$ is*

$$\dim \sigma_s(\nu_{\mathbf{d}}(\mathbb{P}^{\mathbf{1}})) = N - h^0(\mathcal{L}_{m,D}(Z)))$$

where $N = 2^m - 1$ is the dimension of the ambient space $\mathbb{P}(\Bbbk^2 \otimes \ldots \otimes \Bbbk^2)$.

Example 4.3.2 ($\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$)**.** Consider $m = 3$. I.e., let $X = \nu_{1,1,1}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^7$. Since X is a 3-fold we expect that the 2-nd secant variety fills the ambient space. By Corollary 4.3.1, we need to consider the linear system $\mathcal{L}_{3,3}(2^5)$ which we know to be empty by Example 3.4.1.

Example 4.3.3 ($\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is 3-defective)**.** Let $X = \nu_{1,1,1,1}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^{15}$. By the usual parameter count, $\exp.\dim \sigma_3(X) = 3{\cdot}4{+}2 = 14$, hence, we expect $\sigma_3(X)$ to be a hypersurface. By Corollary 4.3.1,

$$\dim \sigma_3(X) = 15 - h^0(\mathcal{L}_{4,4}(3^4, 2^3)).$$

Now, we use the Cremona Reduction Lemma. Since $k = 3\cdot 4 - 3\cdot 4 - 2 = -2$, we have

$$\dim \sigma_3(X) = 15 - h^0(\mathcal{L}_{4,2}(1^4, 2^2)) = 15 - h^0(\mathcal{L}_{2,2}(1^4)),$$

where the latter equality holds because quadrics passing through 2-fat points are cones with such points in the vertex. Now, it is immediate to see that $\dim \sigma_3(X) = 13$ instead of 14.

The latter example was actually proved to be the only case among products of $\mathbb{P}^1$'s.

Theorem 4.3.4 ([CGG11]). *The Segre variety $\nu_{\mathbf{1}}(\mathbb{P}^{\mathbf{1}})$ of n copies of $\mathbb{P}^1$ is never s-defective, unless $(n, s) = (4, 3)$.*

4.4. **On defectiveness of Segre-Veronese varieties.** In [AOP09], a classification of defective Segre varieties is done up to the 6-th secant variety. In general, there is a list of defective cases, see [AOP09].

- $\mathbf{n} = (1, 1, 1, 1)$ and $s = 3$ (Example 4.3.3);
- $\mathbf{n} = (2, 2, 2)$ and $s = 4$;
- $\mathbf{n} = (2, 3, 3)$ and $s = 5$;
- $\mathbf{n} = (2, n, n)$ with $n \in 2\mathbb{N}$ and $s = \frac{3n}{2} + 1$;
- $\mathbf{n} = (1, 1, n, n)$ and $s = 2n + 1$;
- the *unbalanced* cases, i.e., if $\mathbf{n} = (n_1, \ldots, n_m, \bar{n})$ with $\bar{n} > \prod_i(n_i + 1) - \sum_i n_i$ and $\prod_i(n_i + 1) - \sum_i n_i < s \leq \min\{n, \prod_i(n_i + 1) - 1\}$.

Despite the effort, no other defective cases of Segre varieties have been found and it is common belief that these are the only ones. The difficulty in trying to prove that these are the only ones is in finding a winning multiple-induction strategy (on degrees and dimensions), but, even more, in handling the base cases whose proof requires strategies that avoid all defective cases. For example, in the proof of Alexander-Hirschowitz Theorem the case of cubics was particularly complicated and that is why different degeneration techniques have been proposed since then, see Remark 4.1.10.

A similar story holds also for Segre-Veronese varieties. In particular, it has bee considered the case of two factors. Indeed, a list of defective cases for $\nu_{(d,e)}(\mathbb{P}^m \times \mathbb{P}^n)$ is known, see [AB13]. However, all the known defective cases are such that either $d \leq 2$ or $e \leq 2$. If $d \geq 3$ and $e \geq 3$, Abo and Brambilla gave an inductive proof that there are no defective cases assuming that there are none in the base cases $\{(3, 3), (3, 4), (4, 4)\}$. The latter cases were recently solved in [GO22] by using a degeneration technique, which dates back to [Eva97], and consists in collapsing $m + n + 1$ of the general 2-fat base points to have the same support.

Theorem 4.4.1 ([AB13]+[GO22]). *If $d \geq 3$ and $e \geq 3$ then $\nu_{(d,e)}(\mathbb{P}^m \times \mathbb{P}^n)$ is non defective.*

The $(1, e)$ and $(2, e)$ cases are open, in general. See [AB13] for the list of known cases. Recall that the $(1, e)$ case is particularly interesting because related to the study of simultaneous ranks, see Remark 1.1.12.

Lecture V

In the previous lectures, we considered the problem of computing the *general* rank. We want now to address the problem of studying decompositions of given points. Once again we start by considering the most studied case of Waring rank of homogeneous polynomials.

Remark 5.0.2. It is worth to underline that the computation of ranks of special forms is a very complicated task. In particular, in [HL13],it was proved that the computation of the tensor rank is an NP-hard problem and an analogous result for symmetric rank can be found in [Shi16].

5.1. **Apolarity Lemma.** We consider the standard graded polynomial rings $S = \Bbbk[x_0, \ldots, x_n] = \bigoplus_d S_d$ and $R = \Bbbk[y_0, \ldots, y_n] = \bigoplus_d T_d$.

Definition 5.1.1 (Apolarity action). We define the **apolar action** of R on S by partial derivatives. I.e., the extension by linearity of the following relation between monomials:

$$y^\beta \circ x^\alpha = \begin{cases} \frac{\beta!}{(\alpha-\beta)!} x^{\alpha-\beta} & \text{if } \alpha_i \geq \beta_i, \forall i; \\ 0 & \text{otherwise.} \end{cases}$$

Here we use the usual notations where, for any $\alpha = (\alpha_0, \ldots, \alpha_n)$, $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ and $\alpha! = \alpha_0! \cdots \alpha_n!$.

Remark 5.1.2. The definition of the apolar action could have been done also over fields of finite characteristic. In that case however, we would need to consider S to be a *divided powers ring* and we would have considered the simple contraction. E.g., if $x^{[\alpha]} = \frac{1}{\alpha!} x^\alpha$ then $y^\beta \circ x^{[\alpha]} = x^{[\alpha-\beta]}$. See [IK99].

Remark 5.1.3. The apolar action defines a $\Bbbk$-bilinear non-degenerate pairing

$$R_d \times S_d \to \Bbbk, \quad (g, f) \mapsto g \circ f. \tag{5.14}$$

Moreover, if $f = \ell^d = (p_0 x_0 + \ldots + p_n x_n)^d \in S_d$ and $g \in R_d$, then

$$g \circ \ell^d \propto g(p), \qquad \text{where } p = (p_0 : \ldots : p_n). \tag{5.15}$$

Therefore,

$$f \in \langle \ell_1^d, \ldots, \ell_s^d \rangle \quad \text{if and only if} \quad \{g \in R_d \ : \ g \circ \ell_i^d = 0, \ \forall i = 1, \ldots, s\} \subseteq \{g \in R_d \ : \ g \circ f = 0\}.$$

One direction follows directly by (5.15). Viceversa, if $\{g \in R_d \ : \ g \circ \ell_i^d = 0, \ \forall i = 1, \ldots, s\} \subseteq \{g \in R_d \ : \ g \circ f = 0\}$ then, by non-degenericity of the bilinear pairing, the orthogonal spaces are such that

$$\{g \in R_d \ : \ g \circ \ell_i^d = 0, \ \forall i = 1, \ldots, s\}^\perp \supset \{g \in R_d \ : \ g \circ f = 0\}^\perp.$$

The right-hand-side is simply f while the left-hand-side is exactly the linear span of $\langle \ell_1^d, \ldots, \ell_s^d \rangle$.

Definition 5.1.4. Given $f \in S_d$, the **apolar ideal** of f is the annihilator of f with respect to the apolar action. I.e.,

$$\mathrm{Ann}(f) = \{g \in R \ : \ g \circ f = 0\}.$$

Remark 5.1.5. From the bilinear pairing (5.14), the apolar ideal of a given homogenous polynomial $f \in S_d$ can be computed by simple linear algebra degree-by-degree. For any $1 \leq i \leq d$, the *catalecticant map* of f is

$$\mathrm{cat}_i(f) : R_i \to S_{d-i}, \quad g \mapsto g \circ f.$$

Hence, $(\mathrm{Ann}(f))_i = \ker \mathrm{cat}_i(f)$.

Lemma 5.1.6 (Apolarity Lemma)**.** *Let $f \in S_d$ and $Z = \{p_1, \ldots, p_s\}$ be a set of points in $\mathbb{P}^n$. The following are equivalent:*

(1) $f \in \langle \ell_1^d, \ldots, \ell_s^d \rangle$;
(2) $I(Z) \subseteq \mathrm{Ann}(f)$.

Proof. From Remark 5.1.3, we just need to prove that $I(Z)_d \subseteq \mathrm{Ann}(f)$ only if $I(Z) \subseteq \mathrm{Ann}(f)$. Let $g \in I(Z)_e$. If $e > d$, then clearly $g \in \mathrm{Ann}(f)$. Let $e \leq d$. Then, $R_{d-e} g \subseteq I(Z)_d$. By assumption, $y^\alpha g \circ f = 0$ for all $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = d - e$. By non-degenericity of the pairing, we must have $g \circ f = 0$. □

Remark 5.1.7. The apolarity lemma can be rephrased in a more general way by considering 0-dimensional schemes which are not reduced. Let $f \in S_d$ and $Z \subseteq \mathbb{P}^n$ a 0-dimensional scheme. The following are equivalent:

- $f \in \langle \nu_d(Z) \rangle$;
- $I(Z) \subseteq \mathrm{Ann}(f)$.

Moreover, the lemma can be extended to any toric variety by considering a similar apolar action in terms of the Cox ring of the variety. See [GRV18, Gał23].

Remark 5.1.8. Given a homogeneous ideal $I \subset S = \Bbbk[x_0, \ldots, x_n]$, we consider the **Hilbert function**

$$\mathrm{HF}_{S/I}(i) = \dim S_i - \dim I_i.$$

Given a 0-dimensional scheme $Z \subset \mathbb{P}^n$ with defining ideal $I(Z) \subset S$, $\mathrm{HF}_Z(i) = \mathrm{HF}_{S/I(Z)}(i) = \dim S_i - \dim I(Z)_i$. Recall that the Hilbert function of a 0-dimensional scheme in projective space is strictly increasing until it reaches the degree of the scheme and then it stabilizes. See [Chi19, Lemma 2.16] for properties of Hilbert functions of sets of points. It naturally follows that, if Z is a scheme of reduced points computing the rank of $f \in R_d$, then

$$\mathrm{wrk}(f) = \#(Z) \geq \mathrm{HF}_Z(i) \geq \mathrm{HF}_{R/\mathrm{Ann}(f)}(i), \qquad \text{for all } i \geq 0.$$

Example 5.1.9. Consider the monomial $m = x_0 x_1^{d-1}$. The apolar ideal is given by $\mathrm{Ann}(m) = (y_0^2, y_1^d)$. A set of reduced points in $\mathbb{P}^1$ is defined by a completely reducible polynomial. For any degree $2 \leq j \leq d-1$, every polynomial in $\mathrm{Ann}(m)$ is divisible by y_0^2 and, therefore, it is not completely reducible. Therefore, $\mathrm{wrk}(m) \geq d$. On the other hand,the general polynomial $y_0^2 g - y_1^d$ has distinct roots and, therefore, corresponds to a Waring decomposition of m. For example, in the case $d = 2$, the decomposition (1.5) of $x_0 x_1^2$ corresponds to the apolar ideal $(y_0^2 y_1 - y_1^3)$. Hence, $\mathrm{wrk}(m) = d$.

Remark 5.1.10 (Catalecticant method)**.** An algorithm to compute the rank of a given $f \in S_d$ goes as follows.

(1) Compute the catalecticant matrix $\mathrm{cat}_{\lceil d/2 \rceil}(f)$.
(2) Compute the kernel of $K = \ker \mathrm{cat}_{\lceil d/2 \rceil}(f)$.
(3) Let Z be the zero-set of K.
    (a) If Z is not a finite set of points, then the algorithm fails.
    (b) If $Z = \{\ell_1, \ldots, \ell_r\}$ is a reduced set of points, then $\mathrm{wrk}(f) = r$ and by linear algebra we can compute $\lambda_1, \ldots, \lambda_r$ such that $f = \sum_{i=1}^r \lambda_i \ell_i^d$.

It is immediate to see that the quotient algebra $A_f = R/\mathrm{Ann}(f)$ is *artinian*, i.e., a finite dimensional $\mathbb{k}$-vector space: indeed, for any $d > \deg(f)$ we have $R_d = (\mathrm{Ann}(f))_d$. Moreover, as a consequence of the bilinear pairing (5.14), it is also possibile to show that the Hilbert function of A_f is symmetric, see e.g. [Ger96, Proposition 8.6]. In other words, A_f is an *artinian Gorenstein algebra of socle degree d*. It is actually a celebrated result of Macaulay that this is a complete characterization.

Theorem 5.1.11 (Macaulay's classification of artinian Gorenstein graded algebras, see [Ger96, Theorem 8.7])**.** *Let A be a graded artinian Gorenstein algebra with socle degree d. Then, there exists a degree-d homogeneous polynomial $f \in \mathbb{k}[x_0,\ldots,x_n]$ such that $A \cong R/\mathrm{Ann}(f)$.*

5.2. **Sylvester's Algorithm for binary forms.** The Apolarity Lemma gives us an algorithm to compute Waring ranks of binary forms. This method goes back to Sylvester [Syl51].

In the case of binary forms, the apolar ideal has a very complete characterization. Let $S = \mathbb{k}[x_0, x_1]$.

Lemma 5.2.1. *Let $f \in S_d$ be a binary form. Then,* $\mathrm{Ann}(f) = (g_1, g_2)$ *with* $\deg(g_1) + \deg(g_2) = \deg(f) + 2$.

Proof. In codimension two, it is well-known that being Gorenstein is equivalent to being a complete intersection [Ser60]. In particular, if f is a binary form, from Macaulay's Theorem, we have that $\mathrm{Ann}(f)$ is a complete intersection. Since A_f has socle degree equal to d and the Hilbert function of A_f is symmetric, see [Ger96, Proposition 8.6], it is immediate to deduce that $\mathrm{Ann}(f) = (g_1, g_2)$ with $\deg(g_1) + \deg(g_2) = \deg(f) + 2$. □

Therefore, we have the following result which gives also an explicit algorithm to compute the rank (and a Waring decomposition) of any binary form $f \in S_d$.

Theorem 5.2.2. *Let $f \in S_d$ be a binary form such that* $\mathrm{Ann}(f) = (g_1, g_2)$. *Assume* $\deg(g_1) \leq \deg(g_2)$. *Then, the Waring rank of f is:*

- *if g_1 is square free, i.e., it has not multiple roots, then* $\mathrm{wrk}(f) = \deg(g_1)$;
- *otherwise,* $\mathrm{wrk}(f) = \deg(g_2)$.

Proof. By Apolarity Lemma, it follows directly from Lemma 5.2.1. □

Remark 5.2.3 (Sylvester's algorithm for binary forms)**.** Let $f \in S_d$ be a binary form.

(1) Set $r = 1$.
(2) Compute the catalecticant matrix $\mathrm{cat}_r(f)$.
  (a) If $\mathrm{cat}_r(f)$ is of maximal rank, then substitute $r = r + 1$ and come back to (2).
  (b) If $\mathrm{cat}_r(f)$ is not of maximal rank, take a random element $g \in \ker \mathrm{cat}_r(f)$ and go to (3).
(3) Compute the roots of g.
  (a) If g has distinct roots $\{\ell_1,\ldots,\ell_r\}$, then $\mathrm{wrk}(f) = \deg(g)$ and go to (4).
  (b) If g has multiple roots, set $r = d+2-\deg(g)$ and pick a random element of $g' \in \ker \mathrm{cat}_r(f)$. Then, $\mathrm{wrk}(f) = \deg(g')$. Let $\{\ell_1,\ldots,\ell_r\}$ the distinct roots of g' and got to (4).
(4) By linear algebra, compute $\lambda_1,\ldots,\lambda_r$ such that $f = \sum_i \lambda_i \ell_i^d$.

Theorem 5.2.2 was improved to a complete stratification of the space of degree-d binary forms with respect to their ranks [CS11]. Recall the following classical result on the ideals of secants of Rational Normal Curves.

Proposition 5.2.4 ([Har, Theorem 9.7])**.** *Let $\mathcal{C}_d = \nu_d(\mathbb{P}^1)$ be a Rational Normal Curve of $\mathbb{P}^d = \mathbb{P}(S_d)$. Let $\{z_0,\ldots,z_d\}$ be the coordinates on $\mathbb{P}^d$. I.e., the generic element of S_d is $\sum_{i=0}^d \binom{d}{i} z_i x_0^{d-i} x_1^i$. For $r \leq \min\{s, d-s\}$, the r-th secant variety is the rank-r determinantal variety of the catalecticant matrix*

$$\mathrm{cat}_s(f) = \begin{pmatrix} z_0 & z_1 & \cdots & z_{d-s} \\ z_1 & z_2 & \cdots & z_{d-s+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_s & z_{s+1} & \cdots & z_d \end{pmatrix}. \tag{5.16}$$

Assume that $f \in \sigma_r(\mathcal{C}_d) \smallsetminus \sigma_{r-1}(\mathcal{C}_d)$ with $\mathrm{Ann}(f) = (g_1, g_2)$, say $\deg(g_1) \leq \deg(g_2)$. If $f \in \sigma_r(\mathcal{C}_d) \smallsetminus \sigma_{r-1}(\mathcal{C}_d)$, then $\mathrm{cat}_{r-1}(f)$ is maximal rank, while $\mathrm{cat}_r(f)$ is not. In particular, we have that $\deg(g_1) = r$ and, therefore, $\deg(g_2) = d + 2 - r$. Now, by Theorem 5.2.2, we get that either $\mathrm{wrk}(f) = r$ or $\mathrm{wrk}(f) = d + 2 - r$.

Theorem 5.2.5 ([CS11])**.** *Let $\mathcal{C}_d = \nu_d(\mathbb{P}^1)$ be the rational normal curve of $\mathbb{P}^d = \mathbb{P}(S_d)$. Then, for $2 \leq s \leq \lceil \frac{d+1}{2} \rceil$, we have*

$$\sigma_s(\mathcal{C}_d) \smallsetminus \sigma_{s-1}(\mathcal{C}_d) = \sigma_{s,s} \cup \sigma_{s,d-s+2}$$

where $\sigma_{s,i} = \{f \in S_d \ : \ f \in \sigma_s(\mathcal{C}_d), \mathrm{wrk}(f) = i\}$.

After having computed the rank of a form, it is interesting to study *how many* Waring decomposition we have. The space of Waring decompositions of a given form of rank r can be regarded as the fiber of the projection from the abstract r-th secant variety, see Remark 1.1.3.

Definition 5.2.6. Let $f \in S_d$ be any homogeneous polynomials of degree d. Let $r = wrk(f)$. Then, the **Varieties of Sums of Powers** (VSP) of f is

$$\mathrm{VSP}_r(f) = \overline{\{(\ell_1, \ldots, \ell_r) \in \mathbb{P}(S_1)^{\times r} \ : \ f \in \langle \ell_1^d, \ldots, \ell_r^d \rangle\}}.$$

The $\mathrm{VSP}_r(f)$ can also be regarded as a subvariety of the *Hilbert scheme of r points in $\mathbb{P}^n$*. We refer to [RS00] for definitions and the computation of VSP's in several interesting cases.

Once again, in the case of binary forms the answer follows immediately from Theorem 5.2.2.

Proposition 5.2.7. *Let $f \in S_d$ be a degree-d binary form such that $\mathrm{Ann}(f) = (g_1, g_2)$ with $\deg(g_1) \leq \deg(g_2)$. Then:*

(1) *if $\mathrm{wrk}(f) < \lceil \frac{d+1}{2} \rceil$, then there is a unique decomposition, i.e., $\mathrm{VSP}_r(f)$ is a point;*
(2) *if $\mathrm{wrk}(f) \geq \lceil \frac{d+1}{2} \rceil$, then $\mathrm{VSP}_r(f) \cong \mathbb{P}(S_{\deg(g_2)-\deg(g_1)})$.*

Proof. Let $\mathrm{Ann}(f) = (g_1, g_2)$ with $\deg(g_1) \leq \deg(g_2)$. If $r = wrk(f) < \lceil \frac{d+1}{2} \rceil$, then, by Theorem 5.2.2, it means that $r = \deg(g_1) < \lceil \frac{d+1}{2} \rceil \leq \deg(g_2)$ and g_1 is the unique square-free form of degree r. In particular, there is a unique set of r points apolar to f. Otherwise, $r = wrk(f) = \deg(g_2)$ and $\deg(g_1) = d+2-r$. Moreover, g_1 is not square-free and, in particular, there are no square-free forms in $f^\perp$ of degree strictly smaller than $d+2-r$. In degree r, we can consider we have the linear space $\{g_1 h + g_2 \ : \ \deg(h) = \deg(g_2) - \deg(g_1)\} \cong \mathbb{P}(S_{\deg(g_2)-\deg(g_1)})$. The general form $g_1 h + g_2$ is reducible and, in particular, it corresponds to a Waring decomposition of f, i.e., it corresponds to a point of the varieties of sums of powers of f. ☐

Remark 5.2.8. As we have seen, the case of binary forms is completely understood by apolarity theory and Sylvester's catalecticant method. As the reader might expect, with higher number of variables more tools need to be exploited. A very useful algebro-geometric tool is *liaison theory*, that in recent literature proved to be very efficient to understand Waring decompositions of ternary and quaternary forms. This direction would be worth an additional lecture, so, for now, we just refer to the literature. Roughly speaking, liaison theory studies the algebro-geometric properties of two schemes, which are said to be *linked*, whose union is a special schemes, such as a complete intersection or arithmetically Gorenstein, see [PS74, Mig98]. This tool can be used to construct examples of low-dimensional varieties with interesting properties as a link of a simpler one and, at the same time, it can be used to study general objects by linking them to special ones. In the case of Waring decompositions, by Apolarity Lemma, liaison theory is used over sets of reduced points. Liaison theory is quite well undertood for schemes in codimension two and, for this reason, this approach allows to give very good insights on Waring decompositions of ternary forms, see [Mig98, Chapter 6]. For example, it can be used to construct a second Waring decomposition whose corresponding set of points is linked to a given one. This idea can be exploited to deduce *identifiability criteria*, see for example [ACV18, Bal18, ACM19, AC20, AC22] (for definition of identifiability, see Remark 2.4.6); construct forms with special Waring decompositions, see for example [ACO23]; provide a fine stratifications of the forms of given rank by the algebro-geometric properties of their decompositions, see for example [CO23].

5.3. **Waring rank of monomials.** We see now how to use the apolarity lemma to compute the rank of monomials. The following formula was proved by Carlini, Catalisano and Geramita [CCG12]. We recall also the proof because it can provide a nice algebraic strategy to compute lower bounds on the Waring rank of a given form and which is, in general, a very difficult task.

Theorem 5.3.1 ([CCG12])**.** *Let $1 \leq a_0 \leq \ldots \leq a_n$. Let $m = x_0^{a_0} \cdots x_n^{a_n} \in S_d$ be a degree-d monomial. Then,*

$$\mathrm{wrk}(m) = \frac{1}{a_0 + 1} \prod_{i=0}^{n} (a_i + 1).$$

Proof. The apolar ideal of m is $\mathrm{Ann}(m) = (y_0^{a_0+1}, \ldots, y_n^{a_n+1})$. The ideal $(y_1^{a_1+1} - y_0^{a_1+1}, \ldots, y_n^{a_n+1} - y_0^{a_n+1})$ defines a set of points of cardinality $\frac{1}{a_0+1}\prod_{i=0}^n(a_i+1)$. Hence, $\mathrm{wrk}(m) \leq \frac{1}{a_0+1}\prod_{i=0}^n(a_i+1)$.

We prove the opposite inequality. Let Z be a set of points which computes the rank of m. Let $Z' \subseteq Z$ be the subset of points that do not have support on the hyperplane $\{y_0 = 0\}$. In particular, $I(Z') = I(Z) : (y_0)$. Since the Hilbert function of a set of points is strictly increasing until it eventually stabilizes at the cardinality, we get

$$\begin{aligned} \mathrm{wrk}(m) = \#(Z) \geq \#(Z') &= \mathrm{HF}_{R/I(Z')}(t) \\ &= \sum_{i=0}^{t}\left(\mathrm{HF}_{R/I(Z')}(i+1) - \mathrm{HF}_{R/I(Z')}(i)\right), \text{ for } t \gg 0. \end{aligned} \tag{5.17}$$

Now, note that y_0 is a non-zero-divisor for $R/I(Z')$ since the support of Z' is disjoint from $\{y_0 = 0\}$. In particular, the multiplication by y_0 induces the exact sequence

$$0 \to [R/I(Z')]_i \xrightarrow{\cdot y_0} [R/I(Z')]_{i+1} \to [R/(I(Z') + (y_0))]_{i+1} \to 0.$$

Hence, continuing the chain in (5.17), we get that

$$\begin{aligned} \sum_{i=0}^{t}\left(\mathrm{HF}_{R/I(Z')}(i+1) - \mathrm{HF}_{R/I(Z')}(i)\right) &= \sum_{i=0}^{t}\mathrm{HF}_{R/(I(Z')+(y_0))} \\ &\geq \sum_{i=0}^{t}\mathrm{HF}_{R/(\mathrm{Ann}(m):(y_0)+(y_0))}, \end{aligned}$$

where the latter inequality follows from $I(Z') + (y_0) = I(Z) : (y_0) + (y_0) \subseteq \mathrm{Ann}(m) : (y_0) + (y_0)$. Now, it is a direct computation that

$$\mathrm{Ann}(m) : (y_0) + (y_0) = (y_0, y_1^{a_1+1}, \ldots, y_n^{a_n+1})$$

and that, for $t \gg 0$,

$$\sum_{i=0}^{t}\mathrm{HF}_{R/(\mathrm{Ann}(m):(y_0)+(y_0))} = \dim[R/(\mathrm{Ann}(m) : (y_0) + (y_0))] = \frac{1}{a_0+1}\prod_{i=0}^{n}(a_i+1).$$

This concludes the proof. □

In [BBT13], Buczynska, Buczynski and Teitler study the ideal $I(Z) \cap (x_0)$ rather then $I(Z) : (x_0) + (x_0)$ as in [CCG12]. As a result, they characterize *all* minimal Waring decompositions of monomials.

Theorem 5.3.2. [BBT13] *Let $m = x_0^{a_0}\cdots x_n^{a_n}$ with $a_0 \leq \ldots \leq a_m$. If Z is an apolar set of points which computes the Waring rank of m, then $I(Z)$ is a complete intersection of degrees $(a_1+1, \ldots, a_n+1)$.*

Theorem 5.3.3. [BBT13] *Let $m = x_0^{a_0}\cdots x_n^{a_n}$ with $a_0 \leq \ldots \leq a_m$. Let $\{H_{a_1+1,\ldots,a_n+1}(i)\}_{i\geq 0}$ be the Hilbert function of a complete intersection of degrees $(a_1+1, \ldots, a_n+1)$. Let $r = \frac{1}{a_0+1}\prod_{i=0}^n(a_i+1)$. Then, the $\mathrm{VSP}_r(m)$ is irreducible and*

$$\dim \mathrm{VSP}_r(m) = \sum_{i=1}^{n} H_{a_1+1,\ldots,a_n+1}(a_i - a_0).$$

Remark 5.3.4. Note that the set Z' constructed in the proof of Theorem 5.3.1 is forced to coincide with Z which was chosen as an arbitrary set of points computing the rank of m. In particular, the proof tells us even more about the Waring decompositions of the monomial. Indeed, we deduce that all sets of points computing the Waring rank of the monomial m must have support away from $\{y_0 = 0\}$. In [CCO17], the set of linear forms that cannot appear in a minimal Waring decomposition of a given form $f \in S_d$ is called **forbidden locus** of f, denoted by $\mathcal{F}_f$. The latter observation is rephrased as follows in [CCO17, Theorem 3.3]. If $m = x_0^{a_0}\cdots x_n^{a_n}$ with $a_0 = \ldots = a_m < a_{m+1} \leq \ldots \leq a_n$, then

$$\mathcal{F}_m = \{y_0 = \ldots = y_m = 0\}.$$

In [CCO17], forbidden loci of other families of polynomials have been studied. It might be interesting to underline that in all considered cases the forbidden locus was always non-empty, even if it could be as small as possible: the maximal-rank ternary cubic of rank five, which correspond geometrically to the union of a smooth conic and a tangent line, has only one forbidden point which, interestingly enough, corresponds exactly to the tangency point. This observation rises a curious question: is it true that the forbidden locus of any form is non-empty?

Remark 5.3.5. The strategy of the proof of Theorem 5.3.1 has been exploited further to find efficient lower bounds on the Waring rank of homogeneous polynomials in [CCC+18]. In particular, a form $f \in S_d$ is said to be **e-computable** if there exsists an ideal $I \subset R$ and a form $t \in I_e$ such that

$$\mathrm{wrk}(f) = \frac{1}{e}\sum_{i=0}^{\infty}\mathrm{HF}_{R/(\mathrm{Ann}(f):I+(t))}(i).$$

The proof of Theorem 5.3.1 tells us that monomials are always 1-computable. It is worth mentioning that there exists forms that can be 2-computable, but not 1-computable; see [CCC+18, Example 4.15 and Example 4.23].

5.4. **Border rank and cactus rank.** By definition, a point $p \in \mathbb{P}^n$ has X-rank equal to r if there exists a 0-dimensional reduced scheme $Z \subseteq X$ of length r such that $p \in \langle Z \rangle$. Similarly, it has border X-rank equal to r if there exists a one-parameter family of 0-dimensional reduced schemes $Z_t \subseteq X$, with $t \in (0,1]$ of length r such that $p \in \lim_{t\to 0}\langle Z_t \rangle$. Naïvely, one might expect that the latter is equivalent to say that there exists a 0-dimensional scheme $\overline{Z}$ of length r which is the smooth limit of reduced schemes. That is not true: indeed, the limit of the linear spaces $\langle Z_t \rangle$ might be different than the linear span of $\lim_{t\to 0} Z_t$.

Definition 5.4.1. Let X be a projective variety embedded in $\mathbb{P}^n$. For every point $p \in \mathbb{P}^n$, the **cactus X-rank** of p is the smallest length of a 0-dimensional scheme on X whose linear span contains p. I.e.,

$$\mathrm{crk}_X(p) = \min\{r \ : \ \exists Z \subseteq X, \mathrm{length}(Z) = r, \dim(Z) = 0 \text{ s.t. } p \in \langle Z \rangle\}.$$

Definition 5.4.2. Let X be a projective variety embedded in $\mathbb{P}^n$. For every point $p \in \mathbb{P}^n$, the **smoothable X-rank** of p is the smallest length of a 0-dimensional smoothable scheme on X whose linear span contains p. I.e.,

$$\mathrm{srk}_X(p) = \min\left\{r \ : \ \exists \{Z_t\}_{t\in(0,1]} \subseteq X, Z_t \text{ set of } r \text{ points s.t. } p \in \langle \lim_{t\to 0} Z_t \rangle \right\}.$$

For all points $p \in \mathbb{P}^n$, the following relations hold:

$$\underline{\mathrm{rk}}_X(p) \leq \mathrm{srk}_X(p) \leq \mathrm{rk}_X(p); \tag{5.18}$$

$$\mathrm{crk}_X(p) \leq \mathrm{srk}_X(p) \leq \mathrm{rk}_X(p). \tag{5.19}$$

However, cactus and border ranks are not comparable. The cactus rank can be strictly smaller than the border rank [BR13], but also the other way around [BB15].

5.5. **Some example of wild forms.** Consider the latter definitions with respect to Veronese varieties.

In [BB15], Buczynska and Buczynski call **wild form** a homogeneous polynomial $f \in S_d$ such that $\underline{\mathrm{rk}}(f) < \mathrm{srk}(f)$, while they call **tame** the case $\underline{\mathrm{rk}}(f) = \mathrm{srk}(f)$.

Definition 5.5.1. Let $f \in S_d$. We say that f is **concise** if $(\mathrm{Ann}(f))_1 = 0$. Otherwise, we say that f has m **essential variables** if $\mathrm{HF}_{A_f}(1) = m$.

Remark 5.5.2. As mentioned in Remark 5.1.7, if Z is a 0-dimensional scheme, then $I(Z) \subseteq \mathrm{Ann}(f)$ if and only if $f \in \langle \nu_d(Z) \rangle \cong \mathbb{P}^{r-1}$ where $r \leq \mathrm{len}(Z)$. Therefore, it immediately follows that all ranks $\mathrm{crk}(f), \mathrm{srk}(f), \underline{\mathrm{rk}}(f)$ are bounded from below by the number of essential variables of f.

Example 5.5.3 (Quadrics are tame)**.** Let $f \in S_2$ in n essential variables. Then, we know that $\mathrm{wrk}(f) = n$. By (5.18) and (5.19), $\underline{\mathrm{rk}}(f), \mathrm{srk}(f), \mathrm{crk}(f)$ are at most $n = \mathrm{wrk}(f)$. By Remark 5.5.2, we know that the essential variables are a lower bound, therefore $\underline{\mathrm{rk}}(f) = \mathrm{srk}(f) = \mathrm{crk}(f) = \mathrm{wrk}(f) = n$.

Example 5.5.4 (Binary forms are tame)**.** It follows from Theorem 5.2.2, and the fact that every 0-dimensional scheme in $\mathbb{P}^1$ is smoothable, that all binary forms are tame. Assume that $\mathrm{Ann}(f) = (g_1, g_2)$ with $\deg(g_1) \leq \deg(g_2)$. Then, $\underline{\mathrm{rk}}(f), \mathrm{srk}(f), \mathrm{crk}(f)$ are bounded from below by the Hilbert function of A_f whose maximal value is indeed $\deg(g_1)$. Viceversa, g_1 defines a 0-dimensional scheme and then $\mathrm{srk}(f) = \mathrm{crk}(f) = \deg(g_1)$. Moreover, by Proposition 5.2.4, we also have $\underline{\mathrm{rk}}(f) = \deg(g_1)$.

If $Z \subseteq \mathbb{P}^n$ is a 0-dimensional scheme of length r, then its Hilbert function stabilizes at latest at degree $r-1$. The latter occurs when the Hilbert function of Z has the slowest growth, namely when Z is collinear. Therefore, if $r \leq d+1$, then $\langle \nu_d(Z) \rangle \cong \mathbb{P}^{r-1}$, indeed recall that by Apolarity Lemma, $\dim\langle \nu_d(Z) \rangle = \mathrm{HF}_Z(d) - 1$.

Assume $r \leq d+1$. Let $\mathcal{H}_r^{\mathrm{sm}}$ be the smoothable component of the Hilbert scheme of degree r 0-dimensional schemes in $\mathbb{P}^n$. Let $\varphi_d : \mathcal{H}_r^{\mathrm{sm}} \to \mathrm{Gr}(r-1, \mathbb{P}^N)$ be the map sending a smoothable scheme of length r to the $(r-1)$-dimensional linear space $\langle \nu_d(Z) \rangle$. Now, if $I = \mathbb{P}^N \times \mathrm{Gr}(r-1, \mathbb{P}^N)$ is the incidence variety and π_1, π_2 are the two projections, then $A = \pi_1 \pi_2^{-1}(\varphi_d(\mathcal{H}_r^{\mathrm{sm}}))$ is closed and irreducible because $\mathcal{H}_r$ is irreducible. The dense part of the r-th secant variety $\sigma_r^\circ(\nu_d(\mathbb{P}^n)) = \{p \ : \ \mathrm{rk}_X(p) = r\}$ is dense in A by construction. Hence, we can conclude the following result, see [BGI11, Proposition 11] or [BB14, Proposition 2.5].

Theorem 5.5.5 ([BGI11, BB14])**.** *Let $f \in S_d$. If $\underline{\mathrm{rk}}(f) \leq d+1$, then $\underline{\mathrm{rk}}(f) = \mathrm{srk}(f)$.*

Example 5.5.6 (Ternary cubics are tame)**.** By Theorem 3.3.2, we know that the first secant variety of the Veronese variety in the space of ternary cubcis filling the ambient space is $\sigma_4(\nu_3(\mathbb{P}^2))$. Hence, for any $f \in S_3$ we have $\underline{\mathrm{rk}}(f) \leq 4$. Then, we directly apply Theorem 5.5.5.

Example 5.5.7 (Quaternary cubics are tame)**.** In this case, by Alexander-Hirschowitz Theorem the general rank for quaternary cubics is equal to 5. Then, cubics of border rank at most 4 are tame by Theorem 5.5.5. The case of quaternary cubics of border rank 5 needs is treated separately by analysing the possible Hilbert functions encountered, see [BB15, Section 3.6].

Example 5.5.8 (A wild cubic in five variables)**.** In [BB15, Section 4], a first wild case is presented. That is

$$f = x_0^2x_2 + (x_0 + x_1)^2x_3 + x_1^2x_4 \in \Bbbk[x_0, \ldots, x_4].$$

The border rank of f is equal to 5 because

$$f = \lim_{t\to 0} \frac{1}{t}\left[\frac{1}{3}(x_0 + tx_2)^3 - \frac{1}{3}(x_0 + x_1 + tx_3)^3 - \frac{1}{12}(2x_1 - tx_4)^3 - \frac{1}{9}(x_0 - x_1)^3 + \frac{1}{9}(x + 2x_1)^3\right].$$

Assume that there exists a 0-dimensional scheme $Z \subset \mathbb{P}^4$ of length 5 apolar to f. Since f has five essential variables, then

$$\mathrm{HF}_{R/\mathrm{Ann}(f)} : 1\ 5\ 5\ 1\ -;$$

This is a lower bound degree-by-degree for the Hilbert function of Z, since $I_Z \subset \mathrm{Ann}(f)$, and the Hilbert function of Z is bounded by above by the length of Z. Namely,

$$\mathrm{HF}_{R/\mathrm{Ann}(f)}(2) \leq \mathrm{HF}_{R/I_Z}(2) \leq 5 = \mathrm{HF}_{R/\mathrm{Ann}(f)}(2).$$

Then, $(I_Z)_2 = (\mathrm{Ann}(f))_2$. Then, by explicitely computing the apolar ideal of f, it is possible to deduce a contradiction to the fact that the ideal of Z should be saturated, see [BB15, Section 4].

The presence of wild forms makes very difficult the computation of border ranks of homogeneous polynomials. We have to mention a very important recent contribution in this direction. Buczynska and Buczynski proved a *border version* of Apolarity Lemma, see [BB21]. We recall it here just in the framework of Veronese varieties, but it holds a general version for smooth toric varieties embedded via a complete linear system.

Let $h_{r,n}$ be the Hilbert function of r general points in $\mathbb{P}^n$. Consider the multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,n}}_{\mathbb{P}^n}$ of schemes with Hilbert function equal to $h_{r,n}$, see [HS04]. There exists a unique component of $\mathrm{Hilb}^{h_{r,n}}_{\mathbb{P}^n}$ containing ideals of r reduced points with the prescribed Hilbert function. This is called $\mathrm{Slip}_{r,n}$, see [BB21, Section 3].

Theorem 5.5.9 (Border Apolarity Lemma, [BB21, Theorem 3.15])**.** *Let $f \in S_d$ in $n+1$ variables. Then, the following are equivalent:*

(1) $\underline{\mathrm{rk}}(f) \leq r$;
(2) *there exists an ideal $I \in \mathrm{Slip}_{r,n}$ such that $I \subset \mathrm{Ann}(f)$.*

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