BAD LIMITS IN POWER SUM VARIETIES

KRISTIAN RANESTAD

ABSTRACT. A bad limit in the variety of power sums $VSP(f,s)$ of a homogeneous form f of rank s is a scheme Z of length s that is in the closure of apolar schemes to f but is not itself apolar. We show some necessary conditions for a scheme Z to be a bad limit and that $VSP(f,s)$ for general ternary forms of degree 4 or 6 do not have bad limits.

1. Introduction

Let $S = \mathbf{C}[x_0, \dots, x_n]$ and $T = \mathbf{C}[y_0, \dots, y_n]$ be polynomial rings. We view $x_0, \dots, x_n$ and $y_0, \dots, y_n$ as dual bases of dual spaces S_1 and T_1. Interpreting this duality as differentiation, it induces a duality between S_d and T_d for any $d \geq 0$, and a natural pairing

$$S_e \times T_d \to T_{d-e}.$$

For a homogeneous polynomial $f \in T$ let $I_f = \{g \in S | g(f) = 0\} \subset S$, it is called its *apolar ideal*. A linear form $l \in T_1$ interpreted as a point $[l] \in \mathbf{P}^n := \mathbf{P}(T_1)$ naturally has its ideal $I_{[l]} = \oplus_d \{g \in S_d | g(l^d) = 0\} \subset S$. More generally any subscheme $\Gamma \subset \mathbf{P}^n$ has its homogeneous ideal $I_\Gamma \subset S$.

DEFINITION **1.1.** *We say that a subscheme $\Gamma \subset \mathbf{P}^n$ is apolar to f if $I_\Gamma \subset I_f$.*

REMARK **1.2.** Notice that $I_\Gamma \subset I_f$ if and only if $I_{\Gamma,d} \subset I_{f,d}$.

For a homogeneous polynomial $f \in T_d$ of degree d, we define the variety of sums of powers as the closure

$$VSP(f,s) = \overline{\{\{[l_1], \dots, [l_s]\} \in \mathrm{Hilb}_s(\mathbf{P}^n) \mid \exists \lambda_i \in \mathbf{C} : f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d\}}$$

in the Hilbert scheme of the set of power sums presenting f (see [3]). The relation between apolarity and power sums is given by the following duality lemma (see [3]):

LEMMA **1.3.** *Let $l_1, \dots, l_s \in V$ be linear forms. Then $f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d$ for some $\lambda_i \in \mathbf{C}^*$ if and only if $\Gamma = \{[l_1], \dots, [l_s]\} \subset \mathbf{P}^n$ is apolar to f.*

DEFINITION **1.4.** *We say that $[\Gamma] \in VSP(f,s)$ is a bad limit if Γ is not apolar to f.*

So, a bad limit Γ is a scheme of length s which is a limit of apolar schemes of length s without itself being apolar.

LEMMA **1.5.** *Let $f \in T_d$ and assume $[\Gamma] \in VSP(f,s)$ is a bad limit. If every apolar scheme Γ' imposes independent conditions on forms of degree d, then Γ imposes dependant conditions on forms of degree d.*

Proof. Consider a family of schemes $\Gamma_t \in VSP(f,s)$ that are apolar when $t \neq 0$, with limit $\Gamma = \Gamma_0$. Then there is a subspace $U \subset H^0(I_\Gamma(d))$ which is the limit of $H^0(I_{\Gamma_t}(d)), t \neq 0$. Furthermore $\dim U = h^0(I_{\Gamma_t}(d))$ and, since $H^0(I_{\Gamma_t}(d)) \subset I_{f,d}$, also $U \subset I_{f,d}$. If $U = H^0(I_\Gamma(d))$, then Γ is also apolar by Remark 1.2, so the lemma follows. □

To find first examples of bad limits we consider schemes of length s with many points on a line. On forms of degree d, the least number of points on a line that impose dependant conditions is $d+2$. while the similar bound for conics i $2d+2$. So for $s \geq d+2$ and f of rank s, there may be bad limits in $VSP(f,s)$ contained in a line only if $s \geq d+2$.

Indeed, Joachim Jelisiejew, respectively Michal and Grzegorz Kapustka and the author, have shown (yet unpublished):

PROPOSITION **1.6.** *Let f be a quaternary quadric of rank 4, or a quaternary quartic that contains no quadric in its apolar ideal, then there is a divisor of bad limits in $VSP(f,s)$ with $s=4$ and $s=10$, respectively.*

For ternary forms the question of existence of bad limits is open for $d>8$. For $d \leq 8$ the variety $VSP(f,s)$ is finite for a general f, unless $d=2,3,4,6$. For $d=2$ or 3 a bad limit for a general f would have length 3 or 4. But any scheme Γ of length 3 impose independent conditions on quadrics, and any scheme Γ of length 4 impose independent conditions on cubics. So by the above lemma there are no bad limits in these cases.

PROPOSITION **1.7.** *When f is a ternary quartic form with no quadric in its apolar ideal, then $VSP(f,6)$ contains no bad limits.*

Proof. When f is a ternary quartic form that is not apolar to any conic, then the minimal resolution of the apolar ideal I_f has Betti numbers

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 7 & 7 & - \\ - & - & - & - \\ - & - & - & 1 \end{matrix}\ .$$

It is a Pfaffian resolution (see [3, 1]), i.e. the middle syzygy matrix is, after a linear change of generators, a skew symmetric matrix A_f, and the generators are its maximal pfaffians. Since f is not apolar to any conic, no apolar scheme is contained in a conic. The minimal free resolution of the ideal I_Z of a scheme Z of length 6 in $\mathbf{P}^2$ that is not contained in a conic, has Betti numbers (see [3, 2])

$$\begin{matrix} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{matrix}\ .$$

The scheme Z is apolar to f if only if $I_Z \subset I_f$. In this case the resolution of I_Z injects into the resolution of I_f. But this means that, after a linear change of generators of I_f, the matrix A_f has a (3×3)-block of zeros in the upper left corner. We interpret the matrix A_f as a net of alternating 2-forms on a $V = \mathbf{C}^7$:

$$A_f \mapsto v \cdot A_f \cdot v^t, \quad v \in V^*.$$

Let V be the space of cubics in I_f and let U be the space of cubics in I_Z. Then Z is apolar to f if and only if $U \subset V$, in which case the restriction of A_f to $U^\perp \subset V^*$ is zero. Conversely, if A_f vanishes on a 3-space $W \subset V^*$, then there is a 3×4-matrix of linear syzygies among the 4-space of cubics $W^\perp \subset V$. The set

$$D_f(3, V^*) = \{W \subset V^* | A_f|_W = 0\} \subset G(3, V^*)$$

is closed and coincides with the $VSP(f, 6)$ whenever $W^\perp \subset V \subset I_f$ are generators of the ideal of a scheme of length 6 for every $W^\perp \in D_f(3, V^*)$.

The latter is the case unless the cubics of $W^\perp$ have a common factor.

A common factor in a 4-dimensional space of ternary cubics is necessarily linear. Let l be this factor. Then the apolar ideal of $l(f)$ contains four quadratic forms. Therefore $l(f)$ has cactus rank 2 and is annihilated by a linear form l'. But then f contains $l \cdot l'$ in I_f against the assumption. □

Similarly,

Proposition 1.8. *When f is a ternary sextic form and has no cubic form in its apolar ideal, then $VSP(f, 10)$ contains no bad limits.*

Proof. When f has degree 6 and contains no cubic in its apolar ideal I_f, the minimal resolution of the apolar ideal I_f has Betti numbers

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & 9 & 9 & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & 1 \end{matrix} .$$

It is a Pfaffian resolution (see [3, 1]), i.e. the middle syzygy matrix is, after a linear change of generators, a skew symmetric matrix A_f, and the generators are its maximal pfaffians. The minimal free resolution of the ideal I_Z of a scheme Z of length 10 in $\mathbf{P}^2$ that is not contained in a cubic, has Betti numbers (see [3, 2])

$$\begin{matrix} 1 & - & - \\ - & - & - \\ - & - & - \\ - & 5 & 4 \end{matrix} .$$

In notation analogous to the above, let $V \subset S_4$ be the space of quartics in I_f and consider the set

$$D_f(4, V^*) = \{W \subset V^* | A_f|_W = 0\} \subset G(4, V^*).$$

It is closed and coincides with the $VSP(f, 6)$ whenever $W^\perp \subset V \subset I_f(4) \subset S_4$ are generators of the ideal of a scheme of length 10 for every $W \in D_f(4, V^*)$.

The latter is the case unless the quartics of $W^\perp \subset S_4$ have a common factor. A common factor $g \in S$ in a 5-dimensional space of ternary quartics is necessarily linear or quadratic.

If $g \in S_2$, then $g(f)$ is a quartic and $q'g(f) = 0$ for a 5-space of quadratic forms $q' \in S_2$. A ternary quartic form with a 5-space of quadratic forms in its apolar ideal has rank one, i.e. has a pencil of linear forms in its apolar ideal. So $l \cdot g(f) = 0$ for a pencil of linear forms l, and hence f has a pencil of cubic forms in its apolar ideal. This contradicts our assumption.

If $g \in S_1$, then $g(f)$ is a quintic with at least a 5-space of cubic forms in its apolar ideal. But then the set of quadratic partials $S_3(g(f))$ is a 5-dimensional space of quadrics. In particular, there is a quadric form g', such that $g''(S_3(g(q^3))) = 0$. But then $S_3(g''(g(q^3))) = 0$, so $g'' \cdot g$ is a cubic form in I_f. Since I_f contains no cubic forms, this is a contradiction. □

Acknowledgement This work is supported by the Thematic Research Programme "Tensors: geometry, complexity and quantum entanglement", University of Warsaw, Excellence Initiative – Research University and the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021-2023..

REFERENCES

[1] Buchsbaum, D.,Eisenbud, D.: Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. Amer. J. Math **99**,3, (1977) 447-485

[2] Eisenbud, D.: Commutative algebra. With a view toward algebraic geometry. GTM **150** Springer-Verlag, New York, 1995

[3] Ranestad, K., Schreyer, F -O: Varieties of sums of powers, J.reine angew. Math. **525**, (2000) 147-181.

MATEMATISK INSTITUTT, UNIVERSITETET I OSLO, PO BOX 1053, BLINDERN, NO-0316 OSLO, NORWAY