Tensor decompositions and Theoretical computer science

Lecture 4: Laser method

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Recall: Border rank

- Approximate decompositions:

$$\varepsilon^p T + o(\varepsilon^p) = \sum_{a=1}^{r} u_a(\varepsilon) \otimes v_a(\varepsilon) \otimes w_a(\varepsilon)$$

- Here u_a, v_a, w_a have coordinates polynomial in ε
- Can work with border rank $\underline{R}$ and degenerations $\trianglelefteq$ instead of rank R and restrictions $\leq$
- And then take high Kronecker powers to get from $\underline{R}$ to R

Recall: Asymptotic sum inequality

Theorem (Asymptotic sum inequality, Schönhage 81)

$$\underline{R}\Big(\bigoplus_{k=1}^{p} \langle \ell_k, m_k, n_k \rangle\Big) \leq r \Rightarrow \sum_{k=1}^{p} (\ell mn)^{\omega/3} \leq r$$

- Proven by considering high Kronecker powers
- Matrix multiplication tensors actually combine very well if we consider border decompositions
- $\underline{R}(\langle 4, 1, 4 \rangle \oplus \langle 1, 9, 1 \rangle) \leq 17$
- $\omega < 2.55$ and even $\omega < 2.5$ with a more complicated construction
- Much better than known exact decompositions

Laser method: Idea of an intermediate tensor

- Instead of looking for decompositions of MM tensors directly, use an intermediate tensor
$$\langle \ell, m, n \rangle \trianglelefteq T \trianglelefteq E_r$$
- This is not very interesting — can always just talk about $\langle \ell, m, n \rangle$

- Recall Schönhage's theorem
- higher and higher Kronecker powers of $\langle \ell_1, m_1, n_1 \rangle \oplus \langle \ell_2, m_2, n_2 \rangle$ give better and better estimates via their restrictions

$$\binom{N}{K} \odot \left\langle \ell_1^K \ell_2^{N-K}, m_1^K m_2^{N-K}, n_1^K n_2^{N-K} \right\rangle$$

Laser method: Idea of an intermediate tensor

- Use an intermediate tensor

$$T \trianglelefteq E_r$$

$$T^{\boxtimes N} \trianglerighteq Q(N) \odot \langle M(N), M(N), M(N) \rangle$$

- These two relations imply

$$\lim_{N \to \infty} \frac{1}{N} (\log Q(N) + \omega \log M(N)) \leq \log r$$

- What are the properties of a good intermediate tensor?
- Should have low border rank r
- Should have some structure akin to the sum of matrix multiplication tensors which can be exploited

Small Coppersmith—Winograd tensor

$$cw_m = \sum_{i=1}^{m} (e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i + e_i \otimes e_i \otimes e_0) \in \mathbf{k}^{(m+1)\times(m+1)\times(m+1)}$$

- $cw_1 = W$
- cw_m is a symmetric tensor, the corresponding polynomial is $x_0 \sum_{i=1}^{m} x_i^2$
- Alternatively, cw_2 is equivalent to the polynomial xyz
- cw_2 is also isomorphic to the 3×3 permanent considered as a trilinear form on the rows of the matrix

Small Coppersmith—Winograd tensor: border rank

$$\sum_{i=1}^{m}(e_0 + \varepsilon e_i) \otimes (e_0 + \varepsilon e_i) \otimes (e_0 + \varepsilon e_i) =$$

$$= e_0 \otimes e_0 \otimes e_0$$

$$+\varepsilon \left[\left(\sum_i e_i \right) \otimes e_0 \otimes e_0 + e_0 \otimes \left(\sum_i e_i \right) \otimes e_0 + e_0 \otimes e_0 \otimes \left(\sum_i e_i \right) \right]$$

$$+\varepsilon^2 cw_m + \dots$$

Small Coppersmith—Winograd tensor: border rank

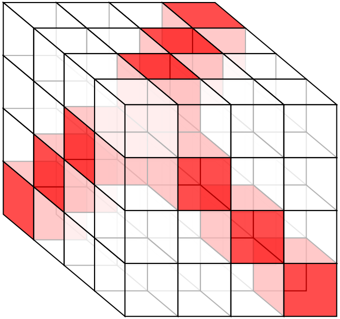
$$\sum_{i=1}^{m} \varepsilon(e_0 + \varepsilon e_i)^{\otimes 3} - \left(e_0 + \varepsilon^2 \left(\sum_i e_i\right)\right)^{\otimes 3} + (1 - m\varepsilon)e_0^{\otimes 3} = \varepsilon^3 cw_m + \ldots$$

$$\underline{R}(cw_m) \leq m + 2$$

- Actually $m + 2$ is the exact value of the border rank for $m \geq 2$

Structure of the Coppersmith—Winograd tensor

$$cw_m = \sum_{i=1}^{m} (e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i + e_i \otimes e_i \otimes e_0) \in \mathbf{k}^{(m+1)\times(m+1)\times(m+1)}$$



Structure of the Coppersmith—Winograd tensor

$$cw_m = \sum_{i=1}^{m} (e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i + e_i \otimes e_i \otimes e_0) \in V \otimes V \otimes V$$

- Decompose $V = k^{m+1}$ into the direct sum

$$V = V_0 \oplus V_1, \quad V_0 = \mathrm{span}(e_0), \quad V_1 = \mathrm{span}(e_1, \dots, e_m)$$

- Let π_0 and π_1 be the corresponding projections onto V_0 and V_1 resp.
- Denote $\pi_{i_1 i_2 i_3} = \pi_{i_1} \otimes \pi_{i_2} \otimes \pi_{i_3}$ and let $T_{i_1 i_2 i_3} = \pi_{i_1 i_2 i_3} cw_m$
- Nonzero blocks:

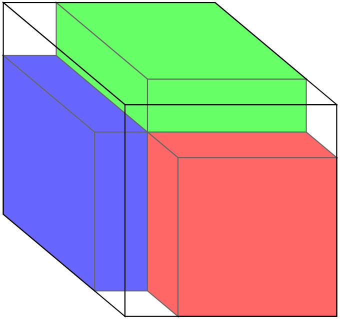
$$T_{101} = \sum_i e_i \otimes e_0 \otimes e_i \cong \langle m, 1, 1 \rangle$$

$$T_{110} = \sum_i e_i \otimes e_i \otimes e_0 \cong \langle 1, m, 1 \rangle$$

$$T_{011} = \sum_i e_0 \otimes e_i \otimes e_i \cong \langle 1, 1, m \rangle$$

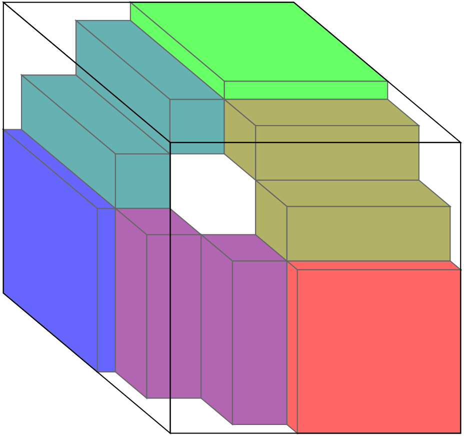
Structure of the Coppersmith—Winograd tensor

$$cw_m = \sum_{i=1}^{m} (e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i + e_i \otimes e_i \otimes e_0) \in \mathbf{k}^{(m+1)\times(m+1)\times(m+1)}$$



Coppersmith—Winograd tensor: Kronecker powers

- Kronecker powers of the CW tensor have an induced block structure
- Each block is the product of original blocks — a matrix multiplication tensor



Coppersmith—Winograd tensor: Kronecker powers

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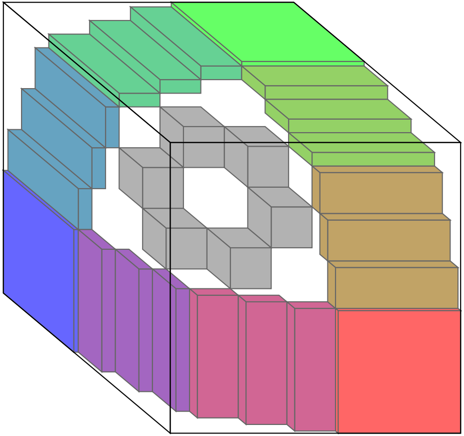

Coppersmith—Winograd tensor: Kronecker powers

- The tensor power space $V^{\otimes N}$ decomposes as

$$V^{\otimes N} = \bigoplus_{i \in \{0,1\}^N} V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_N}$$

- For $i \in \{0,1\}^N$ denote $V_i = V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_N}$ and π_i the projection
- The Kronecker power of cw_m is a (non-direct) sum

$$cw_m^{\boxtimes N} = (\sum_{i \in \{0,1\}^3} T_{i_1 i_2 i_3})^{\boxtimes N} = \sum_{I \in \{0,1\}^{3 \times n}} T_{I_{11} I_{21} I_{31}} \boxtimes T_{I_{12} I_{22} I_{32}} \boxtimes \cdots \boxtimes T_{I_{1N} I_{2N} I_{3N}}$$

- For $I \in \{0,1\}^{3 \times N}$ denote $T_I = T_{I_{11} I_{21} I_{31}} \boxtimes T_{I_{12} I_{22} I_{32}} \boxtimes \cdots \boxtimes T_{I_{1N} I_{2N} I_{3N}}$
- Note that blocks are now indeced by $3 \times N$ matrices of zeroes and ones
- Rows I_1, I_2, I_3 of the matrix I correspond to summands of $V^{\otimes N}$
- The projection $\pi_{I_1} \otimes \pi_{I_2} \otimes \pi_{I_3}$ singles out the block $T_I \in V_{I_1} \otimes V_{I_2} \otimes V_{I_3}$
- Columns $I^1, \dots, I^N$ correspond to the blocks in the original tensor
- $T_I = T_{I^1} \boxtimes T_{I^2} \boxtimes \cdots \boxtimes T_{I^N}$

Laser method: preliminary restriction

- $cw_m^N = \sum_{I \in \{0,1\}^{3 \times N}} T_I$
- $T_I = [\pi_{I_1} \otimes \pi_{I_2} \otimes \pi_{I_3}] cw_m^{\boxtimes N}$
- $T_I = T_{I^1} \boxtimes T_{I^2} \boxtimes \cdots \boxtimes T_{I^N}$
- From now on in all appearing sums we only consider nonzero blocks
- So for all I we assume that the columns are taken from the $\{011, 101, 110\}$

- The first step of the laser method is a preliminary restriction to a subset of blocks of the same "type"
- We say that the row $I \in \{0,1\}^N$ has *type* $(K, N-K)$ if it contains K zeroes and $N-K$ ones
- Let $V_{(K,N-K)}$ be the sum of all V_I with I of type $(K, N-K)$
- From now on assume N is divisible by 3
- Apply to $cw_m^{\boxtimes N}$ the projection π onto the $V_{(\frac{N}{3}, \frac{2N}{3})}$ on all three factors

$$\pi^{\otimes 3} cw_m^{\boxtimes N} = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$$

Laser method: preliminary restriction

- $cw_m^N = \sum_{I \in \{0,1\}^{3\times N}} T_I$
- $T_I = [\pi_{I_1} \otimes \pi_{I_2} \otimes \pi_{I_3}] cw_m^{\boxtimes N}$
- $T_I = T_{I^1} \boxtimes T_{I^2} \boxtimes \cdots \boxtimes T_{I^N}$

- We restricted to a preliminary tensor

$$T^* = \pi^{\otimes 3} cw_m^{\boxtimes N} = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$$

- Since I_1, I_2, I_3 are of type $(N/3, 2N/3)$, they contain $N/3$ zeros each
- This means that column 011 appears $N/3$ times, same for 101 and 110
- Conversely, every block with equal amount of 011, 101 and 110 survive after the restriction
- All blocks in T^* are equivalent to

$$T_{011}^{\boxtimes \frac{N}{3}} \boxtimes T_{101}^{\boxtimes \frac{N}{3}} \boxtimes T_{110}^{\boxtimes \frac{N}{3}} \cong \left\langle m^{\frac{N}{3}}, m^{\frac{N}{3}}, m^{\frac{N}{3}} \right\rangle$$

Laser method: Monomial degeneration

- $T^* = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$
- Each $T_I \cong \left\langle m^{\frac{N}{3}}, m^{\frac{N}{3}}, m^{\frac{N}{3}} \right\rangle$

- In the next step we construct a degeneration of T^* which will eliminate most of the blocks of T_I while keeping some of them
- Most of the remaining blocks will not share their rows in their indexes
- This means that the result will almost be a direct sum of matrix multiplication tensors

- To construct a degeneration, we apply to the k-th factor a linear map which scales the direct summand V_{I_k} by some integer power $\varepsilon^{a_k(I_k)}$
- Then each block T_I is scaled by $\varepsilon^{a(I)}$ where $a(I) = a_1(I_1) + a_2(I_2) + a_3(I_3)$
- We will ensure that $a(I) \geq 0$ by exploiting the property $I_1 + I_2 + I_3 = \vec{2}$
- In the limit $\varepsilon \to 0$ only the blocks with $a(I) = 0$ stay

A simple example of a monomial degeneration

- Here is an example of a monomial degeneration

$$D = \sum_{\sigma \in S_3} e_{\sigma 1} \otimes e_{\sigma 2} \otimes e_{\sigma 3} + e_2 \otimes e_2 \otimes e_2$$

- In each factor scale e_1 and e_3 by ε^{-1} and e_2 by ε^2
- That is, define $A(\varepsilon)(x_1, x_2, x_3) = (\varepsilon^{-1} x_1, \varepsilon^2 x_2, \varepsilon^{-1} x_3)$

$$A(\varepsilon)^{\otimes 3} D = \sum_{\sigma \in S_3} e_{\sigma 1} \otimes e_{\sigma 2} \otimes e_{\sigma 3} + \varepsilon^6 e_2 \otimes e_2 \otimes e_2$$

$$\lim_{\varepsilon \to 0} A(\varepsilon)^{\otimes 3} D = \sum_{\sigma \in S_3} e_{\sigma 1} \otimes e_{\sigma 2} \otimes e_{\sigma 3}$$

Laser method: Monomial degeneration

- $T^* = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$
- Each $T_I \cong \left\langle m^{\frac{N}{3}}, m^{\frac{N}{3}}, m^{\frac{N}{3}} \right\rangle$
- For each of the original block, the sum of indices (011, 101, 110) is 2
- Thus $I_1 + I_2 + I_3 = \vec{2} = (2, 2, \ldots, 2)$

$$I_1 + I_2 = \vec{2} - I_3$$

- Note that if $x + y = z$, then $2x^2 + 2y^2 - z^2 = (x - y)^2 \geq 0$
- Let us convert rows I_1, I_2, I_3 into numbers while keeping the relation intact

Laser method: Hashing

- Let M be a prime number, to be chosen later
- Choose random $\alpha_1, \alpha_2, \ldots, \alpha_N, \beta, \gamma$ iid uniformly in $\mathbb{Z}/M\mathbb{Z}$

$$h_1(I_1) = \sum_{k=1}^{N} \alpha_k I_{1k} + \beta$$

$$h_2(I_2) = \sum_{k=1}^{N} \alpha_k I_{2k} + \gamma$$

$$h_3(I_3) = \sum_{k=1}^{N} \alpha_k (2 - I_{3k}) + \beta + \gamma$$

- We have $h_1(I_1) + h_2(I_2) = h_3(I_3)$ (in $\mathbb{Z}/M\mathbb{Z}$)

- Interpret $h_1(I_1), h_2(I_2), h_3(I_3)$ as numbers in $\{0, 1, \ldots, M-1\}$
- Now there are two possibilities:

$$h_1(I_1) + h_2(I_2) = h_3(I_3)$$

$$h_1(I_1) + h_2(I_2) = h_3(I_3) + M$$

Laser method: Degeneration

- We defined some functions $h_k(I_k)$ depending on random variables α_k, β, γ

$$h_1(I_1) + h_2(I_2) = h_3(I_3)$$

$$\text{or } h_1(I_1) + h_2(I_2) = h_3(I_3) + M$$

- Define

$$a_1(I_1) = 2h_1(I_1)^2, \quad a_2(I_2) = 2h_2(I_2)^2, \quad a_3(I_3) = -h_3(I_3)^2$$

- If $h_1 + h_2 = h_3$, then

$$a_1 + a_2 + a_3 = 2h_1^2 + 2h_2^2 - (h_1 + h_2)^2 = (h_1 - h_2)^2 \geq 0$$

- If $h_1 + h_2 = h_3 + M > h_3$, then

$$a_1 + a_2 + a_3 = 2h_1^2 + 2h_2^2 - (h_1 + h_2 - M)^2 > (h_1 - h_2)^2 \geq 0$$

- The indices for which $a_1 + a_2 + a_3 = 0$ are those where

$$h_1 = h_2 \text{ and } 2h_1 = h_1 + h_2 = h_3 < M$$

Laser method: Degeneration

- $T^* = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$
- $T_I = [\pi_{I_1} \otimes \pi_{I_2} \otimes \pi_{I_3}] T^*$
- Each $T_I \cong \left\langle m^{\frac{N}{3}}, m^{\frac{N}{3}}, m^{\frac{N}{3}} \right\rangle$

$$h_1(I_1) = \sum_{k=1}^{N} \alpha_k I_{1k} + \beta \pmod{M}$$

$$h_2(I_2) = \sum_{k=1}^{N} \alpha_k I_{2k} + \gamma \pmod{M}$$

$$h_3(I_3) = \sum_{k=1}^{N} \alpha_k (2 - I_{3k}) + \beta + \gamma \pmod{M}$$

- We defined a degeneration of T^* which destroys all blocks except those with

$$h_1(I_1) = h_2(I_2) < \frac{M}{2}$$

Laser method: Degeneration

- $T^* = \sum_{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3})} T_I$

$$h_1(I_1) = \sum_{k=1}^{N} \alpha_k I_{1k} + \beta \pmod{M}$$

$$h_2(I_2) = \sum_{k=1}^{N} \alpha_k I_{2k} + \gamma \pmod{M}$$

$$h_3(I_3) = \sum_{k=1}^{N} \alpha_k (2 - I_{3k}) + \beta + \gamma \pmod{M}$$

- We defined a degeneration of T^* which destroys all blocks except those with

$$h_1(I_1) = h_2(I_2) < \frac{M}{2}$$

$$T^{**} = \sum_{\substack{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3}) \\ h_1(I_1) = h_2(I_2) < \frac{M}{2}}} T_I$$

Laser method: Probabilities

$$h_1(I_1) = \sum_{k=1}^{N} \alpha_k I_{1k} + \beta \pmod M$$

$$h_2(I_2) = \sum_{k=1}^{N} \alpha_k I_{2k} + \gamma \pmod M$$

- Let I and J be two indices such that $I_1 = J_1$ and $I \neq J$, so $I_2 \neq J_2$
- $h_2(I_2) - h_2(J_2)$ depends only on α and is uniform as a linear combination of uniform variables in $\mathbb{Z}/M\mathbb{Z}$
- $h_1(I_1)$ and $h_2(I_2)$ are uniform and independent of α because of β and γ summands

$$\mathbb{P}(h_1(I_1) = c_1, h_2(I_2) = c_2) = \frac{1}{M^2}$$

$$\mathbb{P}(h_1(I_1) = c_1, h_2(I_2) = c_2, h_2(J_2) = c_3) = \frac{1}{M^3}$$

Laser method: Probabilities

$$T^{**} = \sum_{\substack{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3}) \\ h_1(I_1) = h_2(I_2) < \frac{M}{2}}} T_I$$

$$\mathbb{P}(h_1(I_1) = c_1, h_2(I_2) = c_2) = 1/M^2$$

$$I_1 = J_1, I \neq J \Rightarrow \mathbb{P}(h_1(I_1) = c_1, h_2(I_2) = c_2, h_2(J_2) = c_3) = 1/M^3$$

- Probability of a block I to remain in T^{**}

$$\mathbb{P}(h_1(I_1) = h_2(I_2) < M/2) = \frac{M}{2}/M^2 = \frac{1}{2M}$$

- Probability for both blocks I and J with $I_1 = J_1$, $I \neq J$

$$\mathbb{P}(h_1(I_1) = h_2(I_2) = h_2(J_2) < M/2) = \frac{M}{2}/M^3 = \frac{1}{2M^2}$$

- Same probability if $I_2 = J_2$ or $I_3 = J_3$

Laser method: Counting

$$T^{**} = \sum_{\substack{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3}) \\ h_1(I_1) = h_2(I_2) < \frac{M}{2}}} T_I$$

$$\mathbb{P}(h_1(I_1) = h_2(I_2) < M/2) = 1/(2M)$$

$$\mathbb{P}(h_1(I_1) = h_2(I_2) = h_2(J_2) < M/2) = 1/(2M^2)$$

- The number of rows of type $(\frac{N}{3}, \frac{2N}{3})$ is $Q = \binom{N}{N/3}$
- If we fix I_1, then the zeros in I_1 determine $\frac{N}{3}$ columns in I containing 011
- The rest $\frac{2N}{3}$ columns are divided between 101 and 110 evenly
- So the number of blocks with fixed I_1 is $P = \binom{2N/3}{N/3}$
- There are $\frac{P(P-1)}{2}$ pairs (I, J) with $I_1 = J_1$, same for $I_2 = J_2$ and $I_3 = J_3$

- The total number of blocks is $QP = \binom{N}{N/3, N/3, N/3}$, expected number of blocks in T^{**} is $\frac{QP}{2M}$
- The total number of pairs with matching indices is $3Q\frac{P(P-1)}{2}$, expected number remaining in T^{**} is $\frac{3QP(P-1)}{4M^2}$

Laser method: Cleanup

$$T^{**} = \sum_{\substack{I_1, I_2, I_3 \text{ of type } (\frac{N}{3}, \frac{2N}{3}) \\ h_1(I_1) = h_2(I_2) < \frac{M}{2}}} T_I$$

- The number of rows of type $(\frac{N}{3}, \frac{2N}{3})$ is $Q = \binom{N}{N/3}$
- So the number of blocks with fixed I_1 is $P = \binom{2N/3}{N/3}$
- The total number of blocks is QP, expected number of blocks in T^{**} is $\frac{QP}{2M}$
- The total number of pairs with matching indices is $3Q\frac{P(P-1)}{2}$, expected number remaining in T^{**} is $< \frac{3QP^2}{4M^2}$

- The last step of the laser method is the removal of all blocks with clashing indices
- Projecting away all V_{I_1} for which there exists at least 2 blocks with $I_1 = J_1$, and the same for 2 and 3
- We remove at most 2 blocks per each clashing pair
- We are left with the independent blocks, the expected number of them is at least $\mathbb{E}[blocks] - 2\mathbb{E}[clashes] = \frac{QP}{2M} - \frac{3QP^2}{2M^2}$ in expectation

Laser method: Parameters

- We degenerated our tensor to a direct sum of blocks, expected number of them is
$$\frac{QP}{2M} - \frac{3QP^2}{2M^2}$$
- Here $Q = \binom{N}{N/3}$ is the number of rows of type $(\frac{N}{3}, \frac{2N}{3})$ is $Q = \binom{N}{N/3}$
- And $P = \binom{2N/3}{N/3}$ is the number of matrices I with a fixed I_1
- M needs to be prime, choose M between $8P$ and $16P$

$$\frac{QP}{2M} - \frac{3QP^2}{2M^2} \geq \frac{Q}{32} - \frac{3Q}{128} = \frac{Q}{128}$$

- For some assignment of random variables, we get at least this number of summands

Laser method: Finale

- We reduced our tensor $cw_m^{\boxtimes N}$ to a direct sum of blocks

$$\frac{1}{128}\binom{N}{N/3} \odot \left\langle m^{N/3}, m^{N/3}, m^{N/3} \right\rangle \trianglelefteq cw_m^{\boxtimes N}$$

$$\frac{1}{128}\binom{N}{N/3} m^{\frac{N}{3}\omega} \leq (m+2)^N$$

$$H(\frac{1}{3}, \frac{2}{3}) + \frac{1}{3}\omega \log m \leq \log(m+2)$$

$$\omega < 2.41 \text{ for } m = 8$$

Laser method: Remarks

- [Coppersmith & Winograd 90] analyzed a more complicated tensor

$$CW_m = \sum_{i=1}^{m} (e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i + e_i \otimes e_i \otimes e_0) +$$

$$+ e_0 \otimes e_0 \otimes e_{m+1} + e_0 \otimes e_{m+1} \otimes e_0 + e_{m+1} \otimes e_0 \otimes e_0$$

- $V = \mathbf{k}^{m+2} = V_0 \oplus V_1 \oplus V_2$
- $V_0 = \mathrm{span}(e_0),\ V_1 = \mathrm{span}(e_1, \ldots, e_m),\ V_2 = \mathrm{span}(e_{m+1})$
- Analysis of the square $CW_6^{\boxtimes 2}$ as a block tensor gives $\omega < 2.376$
- More of less the same method works, but now, instead of using type $(\frac{N}{3}, \frac{2N}{3})$, one needs to optimize over all different types
- In 2010's there was progress related to the analysis of higher powers of CW_m

Laser method: Remarks

- In 2010's there was progress related to the analysis of higher powers of CW_m
- The idea is mostly the same, although there are corrections related to the fact that distribution of columns is not determined by the types of rows for these powers.

[Alman & Vassilevska Williams 2021] $\omega < 2.3728596$