AGATES
ALGEBRAIC GEOMETRY AND COMPLEXITY THEORY WORKSHOP

Plenary Talks

Josh Alman

Title. *Generalizations of Matrix Multiplication can solve the Light Bulb Problem*

Abstract. In the light bulb problem, one is given as input n vectors from $1, 1^d$ which are all uniformly random. They are all chosen independently except for a planted pair which is chosen to have correlation p for some constant $p > 0$, and the goal is to find the planted pair. This problem was introduced over 30 years ago by L. Valiant, and is known to have many applications in data analysis, statistics, and learning theory.

The straightforward algorithm runs in $Omega(n^2)$ time, and algorithms based on Locality-Sensitive Hashing approach quadratic time as p goes to 0. In 2012, G. Valiant gave a breakthrough algorithm that uses fast matrix multiplication and runs in time $O(n^{1.615})$, no matter how small $p > 0$ is. This was subsequently refined by Karppa, Kaski, and Kohonen in 2016 to running time $O(n^{1.582})$, but is essentially the only known approach for this problem.

In this talk, I'll propose a new approach for this problem based on replacing matrix multiplication with other tensors that are similar to matrix multiplication, but may have lower rank (and thus be easier to compute). To demonstrate the potential for this approach, I'll show a $4 \times 4 \times 4$ tensor with rank 5 which is able to solve the problem much faster than using Strassen's algorithm in conjunction with the prior approaches. I'll explain why I'm optimistic about this approach, and more generally, why tensors other than matrix multiplication could lead to faster algorithms for a variety of algorithmic problems. This is based on joint work with Hengjie Zhang.

Markus Bläser

Title. *On the multilinear complexity of associative algebras*

Abstract. Christandl and Zuiddam study the multilinear complexity of d-fold matrix multiplication in the context of quantum communication complexity. Bshouty investigates the multilinear complexity of d-fold multiplication in commutative algebras to understand the size of so-called testers. The study of bilinear complexity is a classical topic in algebraic complexity theory, starting with the work by Strassen. However, there has been no systematic study of the multilinear complexity of multilinear maps. In the present work, we systematically investigate the multilinear complexity of d-fold multiplication in arbitrary associative algebras. We prove a multilinear generalization of the famous Alder-Strassen theorem, which is a lower bound for the bilinear complexity of the (2-fold) multiplication in an associative algebra. We show that the multilinear complexity of the d-fold multiplication has a lower bound of $d \dim A - (d-1)t$, where t is the number of maximal twosided ideals in A. This is optimal in the sense that there are algebras for which this lower bound is tight. Furthermore, we prove the following dichotomy that the quotient algebra $A/\mathrm{rad}A$ determines the complexity of the d-fold multiplication in A: When the semisimple algebra $A/\mathrm{rad}A$ is commutative,

then the multilinear complexity of the d-fold multiplication in A is polynomial in d. On the other hand, when $A/\mathrm{rad}A$ is noncommutative, then the multilinear complexity of the d-fold multiplication in A is exponential in d.

Peter Bürgisser

Title. *Real zeros of polynomials and complexity*

Abstract. Descartes rule from 1687 gives a tight upper bound on the number of real zeros of a univariate polynomial with few monomial terms. Extending this to systems of multivariate polynomials is a long standing open problem. But even univariate polynomials remain mysterious: several conjectures claim that "structured" polynomials should only have few real zeros. These harmless looking conjectures have far reaching consequences: e.g., Koiran's real tau conjecture separates VP from VNP. Hrubes showed that the real tau conjecture can be equivalently restated in terms of the deviation of the angular distribution of complex zeros to the uniform distribution.

Recently, it has been shown that random univariate polynomials typically have as few real zeros as predicted by these conjectures. More concretely, the randomness refers here to independent standard Gaussian coefficients in the arithmetic circuit computing the polynomial, while the combinatorial structure of the circuit is arbitrary and deterministic.

The main technical tool for proving such result is the powerful Kac-Rice formula from probability.

Austin Conner

Title. *Characteristic numbers and chromatic polynomial of a tensor*

Abstract. We describe discrete isomorphism invariants for tensors which generalize and unify the chromatic polynomial of a graph and of a matroid, characteristic numbers of quadrics in Schubert calculus, Betti numbers of complements of hyperplane arrangements, Euler characteristic of complements of determinantal hypersurfaces, and maximum likelihood degree for general linear concentration models in algebraic statistics. Based on joint work with Mateusz Michałek.

Alperen Ergur

Title. *Approximate Rank for Real Symmetric Tensors*

Abstract. We study real symmetric rank of real symmetric tensors with an epsilon room of tolerance for error.

The complex algebraic structure of the problem (without perturbation) is well-studied and is subject to Alexander-Hirschowitz Theorem.

Our aim is to understand/exploit the real geometric nature of the problem, so we use tools from convex geometry and high dimensional probability.

This is joint work with Jesus Rebollo Beuno and Petros Valettas. The link for the arxiv paper and juypter notebook for a preliminary implementation are available on my homepage `https://alpergur.xyz/`

Michael A. Forbes

Title. *Polynomial Identity Testing of Waring Rank Decompositions*

Abstract. Every degree-d (homogeneous) n-variate polynomial can be expressed as a sum of d-th powers of (homogeneous) linear forms, where the number of forms required is known as the Waring rank of that polynomial. In general such rank can be exponential in the relevant parameters, but when it is small we may hope to use the given decomposition to more efficiently manipulate and understand the polynomial the decomposition expresses. The most basic such question is the polynomial identity testing (PIT) question, which asks whether a given algebraic expression (in this case a sum of powers of linear forms) simplifies to the zero polynomial.

In this talk we will discuss what is known about PIT of Waring rank decompositions. In particular, as the Schwartz-Zippel lemma implies a simple randomized algorithms for PIT, the challenge is to develop efficient deterministic algorithms, despite the exponentially many possible monomials that might appear in the fully expanded polynomial. It turns out that Waring rank decompositions are one of the first non-trivial settings of PIT, and substantial work in the past 20 years has nearly, but not quite, resolved the PIT problem for this setting. This talk will survey the known results, in both the white- and black-box settings, as well as highlighting connections other models of algebraic computation including read-once (oblivious) algebraic branching programs.

Christian Ikenmeyer

Title. *Homogeneous algebraic computation*

Abstract. The closure of the complexity class VBP is described via approximations of homogeneous projections of the iterated matrix multiplication polynomial. This enables the study of VBP with methods from geometric complexity theory without having to rely on the padding of polynomials. In an analogous spirit we give a simpler description for a subclass of VBP that is equivalent to VBP up to quasipolynomial blowup. This is related to Kumar's recent work on border complexity.

Gorav Jindal

Title. *On the Complexity of Symmetric Polynomials*

Abstract. The fundamental theorem of symmetric polynomials states that for a symmetric polynomial f_{Sym} in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, there exists a unique "witness" f in $\mathbb{C}[y_1, y_2, \ldots, y_n]$ such that $f_{Sym} = f(e_1, e_2, \ldots, e_n)$, where the e_i's are the elementary symmetric polynomials. In this work, we study the arithmetic complexity $L(f)$ of the witness f as a function of the arithmetic complexity $L(f_{Sym})$ of f_{Sym}. We show that the arithmetic complexity $L(f)$ of f is bounded by $poly(L(f_{Sym}), \deg(f), n)$. Prior to this work, only exponential upper bounds were known for $L(f)$. The main ingredient in our result is an algebraic analogue of Newtons iteration on power series. As a corollary of this result, we show that if VP is not equal to VNP then there are symmetric polynomial families which have super-polynomial arithmetic complexity. This is joint work with Markus Bläser.

Rima Khouja

Title. *Simultaneous matrix diagonalization algorithm for the tensor rank approximation problem*

Abstract. In this talk, we address the connection between simultaneous matrix diagonalization and tensor decomposition. A pencil of matrices $M = [M_1, \ldots, M_s]$ is called in this talk *simultaneously diagonalizable*, if there exists two invertible matrices E and F such that $\Sigma_i := FM_kE$ is a diagonal matrix, for $k \in \{1, \ldots, s\}$. In the first part, we assume that the pencil of matrices is simultaneously diagonalizable, and we construct a Newton-type sequence that converges quadratically towards the solution $(E, F, (\Sigma_i)_{1\leq i\leq s})$. Moreover, we exhibit a certification test that the sequence converges towards the solution. In the second part, we consider that the pencil of matrices is not necessarily simultaneously diagonalizable. Thus, we aim to approximate it locally by a pencil of matrices which is simultaneously diagonalizable. We study this problem from an optimization point of view by taking into account the geometric constraints of the optimization problem and we present a Riemannian conjugate gradient algorithm. We consider the approximation problem over the real field $\mathbb{R}$. Given a general pencil $M = [M_1, \ldots, M_s]$ of real square matrices, such that $M_k \in \mathbb{R}^{n\times n}, \forall k \in \{1, \ldots, s\}$, we aim to compute a simultaneously diagonalizable pencil that approximates M i.e. to find two invertible matrices E and F such that FM_kE^t is the most diagonal, $\forall k \in \{1, \ldots, s\}$. Finally, for real tensor rank approximation problem of real multilinear tensors of dimension 3 and size (n_1, n_2, n_3) with approximation rank $r \geq \max(n_1, n_2)$, we develop an *alternate optimization algorithm*, based on approximate simultaneous diagonalization of matrices. This is a joint work with Bernard Mourrain and Jean-Claude Yacoubsohn.

Bernard Mourrain

Title. *Tensor extensions and decompositions*

Abstract. Tensor decomposition generalizes matrix rank decomposition, but with a much higher degree of complexity. This problem has many fascinating facets both from the algebraic, geometric and application point of view. In this talk, we will describe an approach based on tensor extensions for their decomposition. We will connect to it to standard linear algebra operations such has eigenvector computation, simultaneous diagonalisation of matrix pencils, provide criteria for the existence of isorank extensions and analyse some of their geometric properties. Links with the varieties of commuting matrices and the Hilbert scheme of points will be discussed.

Greta Panova

Title. *Complexity and Algebraic Combinatorics*

Abstract. The complexity of computing polynomials can be understood via their algebraic and geometric properties and symmetries. Geometric Complexity Theory (GCT) translates these problems to Representation Theory, whereby inequalities between multiplicities of irreducible representations could give computational lower bounds. Such multiplicities are also a subject of study of Alge- braic Combinatorics, where the classical but mysterious Kronecker and plethysm coefficients pose some of the main open problems. In this talk we will define the main objects and discuss these connections.

Philipp Reichenbach

Title. *Barriers for Tensor Scaling*

Abstract. In recent years, Computational Invariant Theory has seen significant progress in optimization techniques: scaling algorithms seek to approximately minimize the norm along an orbit and also give rise to deciding null-cone membership. Both computational problems have versatile applications in mathematics, physics, statistics and computer science. In this talk we focus on the natural action of $SL(n)^d$ on d-tensors. For $d = 2$, this leads to operator scaling (or non-rational identity testing), which enjoyed several success stories. There are scaling algorithms that provide high-precision solutions in polynomial time and decide null-cone membership in polynomial time. However, in both counts current techniques are only known to run in exponential time for tensor scaling – that is $d = 3$. We explain this dichotomy by providing bounds on certain complexity parameters. These barriers suggest that current geodesic methods do not suffice for efficient tensor scaling and motivate the search for new algorithms. Based on joint work with Cole Franks, see `arXiv:2102.06652`.

Tim Seynnaeve

Title. *Bounds on complexity of matrix multiplication away from Coppersmith-Winograd tensors*

Abstract. Determining the complexity of matrix multiplication is a central problem in computer science. Its algebraic counterpart translates to estimating the rank or border rank of the matrix multiplication tensor. The complexity of matrix multiplication is measured by the constant ω, defined as the smallest number such that for any $\epsilon > 0$ the multiplication of $n \times n$ matrices can be performed in $O(n^{\omega+\epsilon})$ arithmetic operations. Equivalently, ω is the smallest number such that for any $\epsilon > 0$ the rank (or border rank) of the $n \times n \times n$ matrix multiplication tensor is $O(n^{\omega+\epsilon})$.

It turns out that it is useful to, instead of studying the matrix multiplication tensor directly, consider a different tensor which can be proven to have low border rank, and at the same time is "close" to being a matrix multiplication tensor. This leads to the so-called laser method. The best known upper bounds on ω are all obtained using the laser method to a very specific tensor, known as the Coppersmith-Winograd tensor. However, several recent results proved the existence of barriers: as a very particular case, using the Coppersmith-Winograd tensors alone in the laser method, one cannot obtain an upper bound for ω close to 2.

One approach to breaking these barriers is to find new tensors that are suitable for the laser method. In this talk I will suggest several approaches to constructing such tensors, based on highest weight vectors, smoothable algebras, and monomial algebras.

This talk is based on joint work with Roser Homs, Joachim Jelisiejew, and Mateusz Michalek.

Simon Telen

Title. *Complexity of tensor decomposition and Hilbert functions*

Abstract. Low rank tensor decomposition can be interpreted as solving a system of polynomial equations on products of projective spaces. I will discuss a recent approach based on normal forms whose complexity is governed by the regularity of a non-saturated homogeneous

ideal. That is, it depends on the degree at which the Hilbert function stabilizes. I will also discuss the symmetric case. This is based on joint work with Nick Vannieuwenhoven and discussions with Fulvio Gesmundo.

André Uschmajew

Title. *Maximum relative distance between real rank-two and rank-one tensors*

Abstract. The distance of a tensor to the set of rank-one tensors is related to the ratio of its spectral and Frobenius norm. The minimal possible ratio of these norms measures the largest possible (relative) distance, and is called the best rank-one approximation ratio of the tensor space. Due to the importance of rank-one tensors this quantity is relevant from several perspectives. In the talk, we review this problem and present recent results on the minimal norm ratio for real rank-two tensors. This is joint work with H. Eisenmann.

Ke Ye

Title. *Certifying nonnegative functions on finite sets via Fourier SOS*

Abstract. This talk consists of two parts. In the first part, we introduce a framework of certifying the non-negativity of a function on a finite set, which is equivalent to certifying the nonnegativity of tensors. Via the Fourier sum of squares (FSOS), we are able to discuss both the theoretical and algorithmic aspects of this problem. The second part is concerned with applications of our framework to MAX-SAT. We will discuss how to find low degree FSOS certificates and estimate the lower bound for MAX-SAT problems.

Contributed Talks

Andreas Kretschmer

Title. *When are (radicals of) symmetric ideals monomial?*

Abstract. Given a set of multivariate polynomials, is there anything we can say about the common zeros of all permutations of these polynomials? How hard is it to solve such a symmetric system of polynomials and is this even a rigorous question? A surprising experimental observation is that for general (or: random) coefficients of a polynomial, the set of common solutions to all its permutations is often very simple. I will try to make this intuition precise in the talk and discuss some results relating this to the classical representation theory of the symmetric groups.

Julia Lindberg

Title. *The symmetric geometric rank of symmetric tensors*

Abstract. Inspired by recent work of Kopparty, Moshkovitz and Zuiddam on the geometric rank of a tensor, we define the symmetric geometric rank of a symmetric tensor. In this talk I will outline the notion and basic properties of the symmetric geometric rank of a symmetric tensor. Time permitting, I will discuss numerical methods to compute this. This is joint work with Pierpaola Santarsiero and Jose Rodriguez.

Harold Nieuwboer

Title. *The minimal canonical form of a tensor network*

Abstract. Tensor networks are capable of describing most states of interest in condensed matter physics, while at the same time being sufficiently simple to allow for numerical and theoretical manipulations. Tensor network representations of states have virtual gauge degree of freedoms which do not change the resulting physical state, so having a canonical form is both theoretically and numerically desirable. We define a new normal form for one-dimensional tensor networks, introduce the first canonical form for (uniform) higher dimensional tensor networks which applies to general projected entangled pair states, and give algorithms to compute this normal form. To achieve this we draw on geometric invariant theory and recent progress in theoretical computer science in non-commutative group optimization, with the key player being a simultaneous conjugation action of products of general linear groups.

Based on joint work with Arturo Acuaviva, Visu Makam, David Perez-Garcia, Friedrich Sittner, Michael Walter and Freek Witteveen; full version available at `arXiv:2209.14358`.

Filip Rupniewski

Title. *Direct sum of two copies of 2×2 matrix multiplication tensors has a rank additivity property*

Abstract. We address the problem of the additivity of the tensor rank. That is for two independent tensors we study if the rank of their direct sum is equal to the sum of their individual ranks. A positive answer to this problem was previously known as Strassens conjecture (1969) until 2019, when Shitov proposed counterexamples. The latter are not explicit, and only known to exist asymptotically for very large tensor spaces. I give few condition for the additivity of three-way tensors. As a consequence we obtain that another conjecture by Strassen (1969) is true. This is the special case of the former one. The pair of 2×2 matrix multiplication tensors has the additivity property. Thus, there is no faster way to multiply two pairs of 2×2 matrices, than to multiply the first pair and then the second one independently.