Real aspects of the problem of rank-one approximation

Khazhgali Kozhasov (TU Braunschweig)

AGATES Kickoff Workshop, Warsaw, 19-23 September 2022

Motivation

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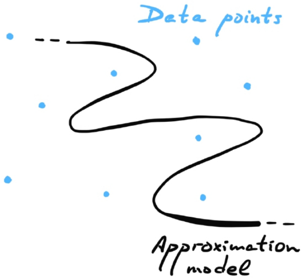

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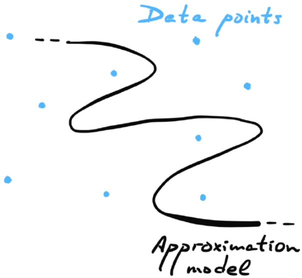

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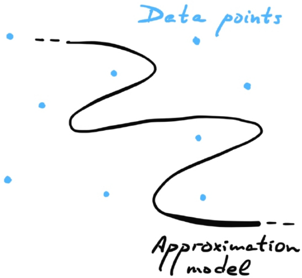

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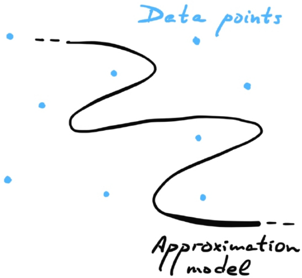

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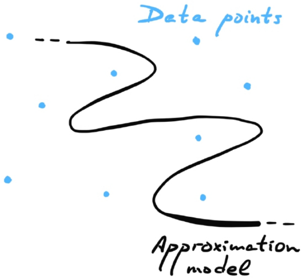

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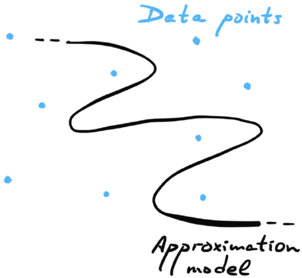

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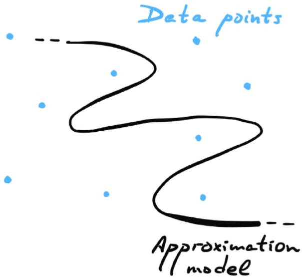

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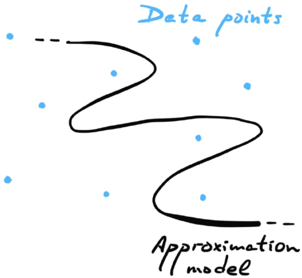

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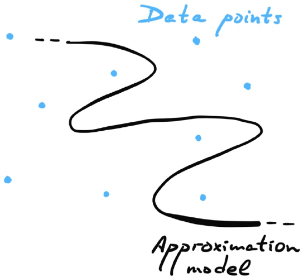

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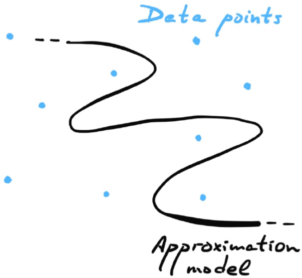

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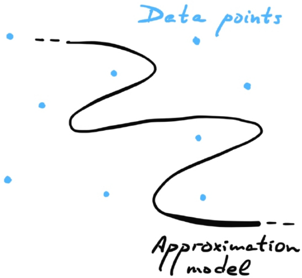

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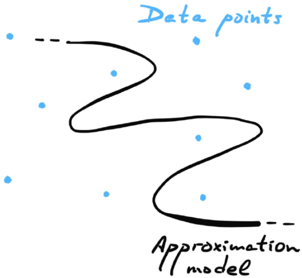

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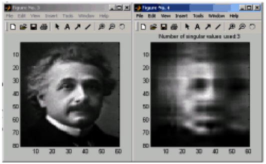

Part 1.
Rank-one approximations of general tensors

Framework and problem statement

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Remark: NP-hard to decide if T^* is a solution, already for $d = 3$.

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Mirsky, 1960: T^* is a solution for a $O(n_1) \times O(n_2)$-invariant norm.

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A best rank-1 approximation to T minimizes the (squared) distance

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Algebraic geometer: study complex solutions of the system (*).

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The result of Friedland and Ottaviani gives a formula for the ED degree $\mathrm{EDdeg}(\mathcal{S}_{\boldsymbol{n}}) = \mathrm{sv}(\boldsymbol{n})$ of the Segre variety $\mathcal{S}_{\boldsymbol{n}} \subset \mathbb{R}^{\boldsymbol{n}}$.

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$\boldsymbol{n} = (n_1, \ldots, n_d)$	$\mathrm{sv}(\boldsymbol{n})$
(n_1, n_2)	$\min(n_1, n_2)$
$2^d = (2, \ldots, 2)$	$d!$
$(2, 2, n \geq 3)$	8
$(2, 3, 3)$	15
$(2, 3, n \geq 4)$	18
$(3, 3, 3)$	37
$(3, 3, 4)$	55
$(3, 3, n \geq 5)$	61
$n_d \geq 1 + \sum_{j=1}^{d-1}(n_j - 1)$	stabilizes

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$$\mathcal{F}(\theta_1, \ldots, \theta_d) = \sum_{\varepsilon \in \{-1,1\}^d} (\chi_\varepsilon \cos(\varepsilon_1\theta_1 + \cdots + \varepsilon_d\theta_d) + \xi_\varepsilon \sin(\varepsilon_1\theta_1 + \cdots + \varepsilon_d\theta_d))$$

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Part 2.
Rank-one approximations of symmetric tensors

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Remark: (still) NP-hard to decide if T^* is a solution, when $d \geq 3$.

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$$(\Delta f)(\mathrm{x}) \; = \; \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \; = \; 0, \quad \mathrm{x} \in \mathbb{R}^n.$$

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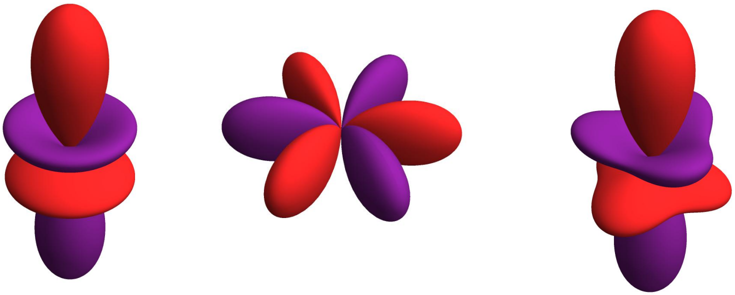
$$\begin{aligned} 2 + 2\,\mathrm{ev}(d,n)\,(d-1) \;&=\; 2\left(1 + \frac{(d-1)^n - 1}{d-2}(d-1)\right) \\ &=\; 2\frac{(d-1)^{n+1} - 1}{d-2} \;=\; 2\,\mathrm{ev}(d,n+1) \end{aligned}$$

critical points on the n-dimensional sphere $\mathbb{S}^n$.

Illustrative example (spherical plot)



$Z_{3,3}$, a harmonic form $f_{3,2}$ with $2\,\mathrm{ev}(3,2) = 6$ critical points on $\mathbb{S}^1$ and the perturbation $f_{3,3} = Z_{3,3} + \varepsilon\, f_{3,2}$ with $2\,\mathrm{ev}_{3,3} = 14$ critical points on $\mathbb{S}^2$.

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How large can numbers $I_0(f)$, $I_1(f)$, $\ldots$, $I_{n-1}(f)$ be for a generic form $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d?

Part 3.
Quality of rank-one approximations

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$\mathcal{A}_{d,n}$ governs the convergence rate of greedy rank-one update algorithm. Was introduced in this context by Qi in 2011.

$$T^{(0)} = T, \quad T^{(k+1)} = T^{(k)} - T_1^{(k)}$$

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$$\frac{\|\mathsf{T} - \mathsf{T}_*\|}{\|\mathsf{T}\|} = \sqrt{1 - \frac{\|\mathsf{T}\|_\infty^2}{\|\mathsf{T}\|^2}}$$

among all symmetric tensor in $\mathrm{Sym}^d(\mathbb{R}^n)$.

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IMPORTANT COROLLARY:

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$$\frac{1}{\sqrt{n^{d-1}}} \leq \mathcal{A}^{\mathrm{sym}}_{d,n} \leq \mathbb{E}_{\mathsf{T} \in \mathrm{Sym}^d_n} \frac{\|\mathsf{T}\|_\infty}{\|\mathsf{T}\|} \leq \frac{12\sqrt{n \ln d}}{\sqrt{\binom{n+d-1}{d}}} \leq 12\sqrt{\frac{d! \ln d}{n^{d-1}}}$$

In particular, for a fixed $d \geq 3$ and $n \to \infty$

$$\mathcal{A}^{\mathrm{sym}}_{d,n} = \mathcal{O}\left(\frac{1}{\sqrt{n^{d-1}}}\right) = \mathcal{A}_{d,n}.$$

IMPORTANT COROLLARY:

Symmetric tensors are as far from being rank-1

Qualitative bounds (symmetric case)

Theorem (K., Tonelli-Cueto, 2022+)

For all $d \geq 3$ and $n \geq 2$ we have

$$\frac{1}{\sqrt{n^{d-1}}} \leq \mathcal{A}_{d,n}^{\mathrm{sym}} \leq \mathbb{E}_{\mathsf{T} \in \mathrm{Sym}_n^d} \frac{\|\mathsf{T}\|_\infty}{\|\mathsf{T}\|} \leq \frac{12\sqrt{n \ln d}}{\sqrt{\binom{n+d-1}{d}}} \leq 12\sqrt{\frac{d! \ln d}{n^{d-1}}}$$

In particular, for a fixed $d \geq 3$ and $n \to \infty$

$$\mathcal{A}_{d,n}^{\mathrm{sym}} = \mathcal{O}\left(\frac{1}{\sqrt{n^{d-1}}}\right) = \mathcal{A}_{d,n}.$$

IMPORTANT COROLLARY:

Symmetric tensors are as far from being rank-1 as general tensors.

Thank you!