Algebraic Geometry with Applications to TEnsors and Secants – Kickoff workshop
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De-bordering symmetric border rank

–and other open problems–

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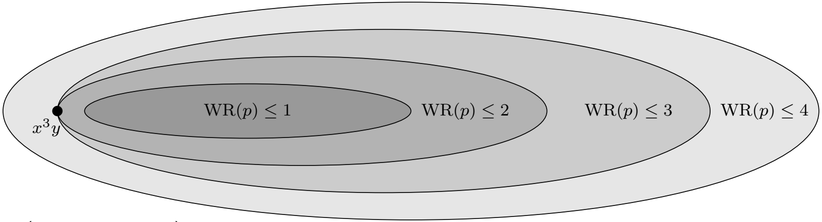
For a homogeneous degree d polynomial p define the **Waring rank** $\mathrm{WR}(p)$ (also called **symmetric rank**) as the smallest r such that there exist homogeneous linear polynomials with

$$p = \sum_{i=1}^{r} (\ell_i)^d.$$

$12x^3y = (x+y)^4 + i^3(x+iy)^4 + i^2(x+i^2y)^4 + i(x+i^3y)^4$, hence $\mathrm{WR}(x^3y) \le 4$. In fact, $\mathrm{WR}(x^{d-1}y) = d$.



$\frac{1}{\varepsilon}\Big((x+\varepsilon y)^4 - x^4\Big) = 4x^3y + \varepsilon(6x^2y^2 + 4\varepsilon xy^3 + \varepsilon^2 y^4) \xrightarrow{\varepsilon\to 0} 4x^3y$

The **border Waring rank** $\underline{\mathrm{WR}}(p)$ is defined as the smallest r such that p can be approximated arbitrarily closely by polynomials of Waring rank $\le r$. For example, $\underline{\mathrm{WR}}(x^3y) \le 2$.

In fact [Carlini, Catalisano, Geramita 2012]: $\mathrm{WR}(x^{d-1}y) = d$, whereas $\underline{\mathrm{WR}}(x^{d-1}y) = 2$.

Theorem (works in high generality)

Let $V = \mathbb{C}[x_1, \dots, x_n]_d$. Zariski closure and Euclidean closure coincide:

$$\{p \in V \mid \underline{\mathrm{WR}}(p)\} = \overline{\{p \in V \mid \mathrm{WR}(p)\}}^{\mathbb{C}} = \overline{\{p \in V \mid \mathrm{WR}(p)\}}^{\mathsf{Zar}}.$$

(secant variety of the Veronese variety)

Analogously: The Chow rank

For a homogeneous degree d polynomial p define the **Chow rank** $\mathrm{CR}(p)$ as the smallest r such that there exist homogeneous linear polynomials $\ell_{i,j}$ with

$$p = \sum_{i=1}^{r} \prod_{j=1}^{d} \ell_{i,j}.$$

The **border Chow rank** $\underline{\mathrm{CR}}(p)$ is defined as the smallest r such that p can be approximated arbitrarily closely by polynomials of Chow rank $\leq r$.

Analogous theorem

Let $V = \mathbb{C}[x_1, \ldots, x_n]_d$. Zariski closure and Euclidean closure coincide:

$$\{p \in V \mid \underline{\mathrm{CR}}(p)\} = \overline{\{p \in V \mid \mathrm{CR}(p)\}}^{\mathbb{C}} = \overline{\{p \in V \mid \mathrm{CR}(p)\}}^{\mathsf{Zar}}.$$

(secant variety of the Chow variety (i.e., variety of products of linear forms))

For a homogeneous degree d polynomial p the **Waring rank** $\mathrm{WR}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \cdots \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_r \end{pmatrix}$$

For a homogeneous degree d polynomial p the **Chow rank** $\mathrm{CR}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,2} \end{pmatrix} \begin{pmatrix} \ell_{1,3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,d-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,d-1} \end{pmatrix} \begin{pmatrix} \ell_{1,d} \\ \vdots \\ \ell_{r,d} \end{pmatrix}$$

For a homogeneous degree d polynomial p the **complexity** $\mathrm{w}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

This is also called the **iterated matrix multiplication complexity** or the **algebraic branching program width**.

$\underline{\mathrm{w}}$ is defined analogously to $\underline{\mathrm{WR}}$.

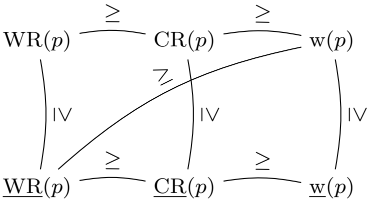
Recall

For a homogeneous degree d polynomial p the **complexity** $\mathrm{w}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix}\ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1}\end{pmatrix} \begin{pmatrix}\ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2}\end{pmatrix} \begin{pmatrix}\ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3}\end{pmatrix} \cdots \begin{pmatrix}\ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1}\end{pmatrix} \begin{pmatrix}\ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d}\end{pmatrix}$$

$$\begin{array}{ccccc} \mathrm{WR}(p) & \geq & \mathrm{CR}(p) & \geq & \mathrm{w}(p) \\ \mathrel{\rotatebox{90}{$\leq$}} & & \mathrel{\rotatebox{90}{$\leq$}} & & \mathrel{\rotatebox{90}{$\leq$}} \\ \underline{\mathrm{WR}}(p) & \geq & \underline{\mathrm{CR}}(p) & \geq & \underline{\mathrm{w}}(p) \end{array}$$

- For $p = x^2y$ we have $\mathrm{WR}(p) = 3 > 2 = \underline{\mathrm{WR}}(p)$
- For $p = y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 + x_1x_2x_3$ we have $\mathrm{CR}(p) > 2 = \underline{\mathrm{CR}}(p)$ [Hüttenhain 2017]
- Gesmundo's remark: For $p = (x_1y_1 + \cdots x_8y_8)z^{39}$ we have $\mathrm{w}(p) > 2 = \underline{\mathrm{w}}(p)$ proved via combining [Allender-Wang 2015] and [Bringmann, I, Zuiddam 2017]



Theorem (de-bordering) [Bläser, Dörfler, I 2020]

$\underline{\mathrm{WR}}(p) \geq \mathrm{w}(p)$.

Proof: Via Nisan's flattenings ("tensor partial derivatives").

De-bordering is a recent topic in algebraic complexity theory. New techniques in [Dutta Dwivedi Saxena 2022].

Linear forms $a_{i,j}$ and $b_{i,j}$.

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,r} \end{pmatrix} \boxtimes \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} \\ \vdots & \ddots & \vdots \\ b_{r,1} & \cdots & b_{r,r} \end{pmatrix} := \begin{pmatrix} \sum_{i=1}^r a_{1,i} \otimes b_{i,1} & \cdots & \sum_{i=1}^r a_{1,i} \otimes b_{i,r} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^r a_{r,i} \otimes b_{i,1} & \cdots & \sum_{i=1}^r a_{r,i} \otimes b_{i,r} \end{pmatrix}$$

Definition

For an order d tensor t the **tensor complexity** $\mathrm{w}_\otimes(t)$ is defined as the smallest r such that $\exists$ linear forms with $t =$

$$\begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

Theorem [Nisan 1991]

Consider the maps F_i:

$$\begin{array}{ccc} \bigotimes^d V & \simeq & \left(\bigotimes^i V\right) \otimes \left(\bigotimes^{d-i} V\right). \\ t & \mapsto & F_i(t) \end{array}$$

We have $\forall t: \ \mathrm{w}_\otimes(t) = \max\{\mathrm{rank}(F_i(t))\}$.

Nisan's proof is combinat. + lin. alg. (in "Lower Bounds for Non-Commutative Computation"). Basis-independent version?

Conclusion [Forbes 2016]

$\forall t: \ \underline{\mathrm{w}_\otimes}(t) = \mathrm{w}_\otimes(t)$.

Theorem (de-bordering) [Bläser, Dörfler, I 2020]

$\underline{\mathrm{WR}}(p) \geq \mathrm{w}(p)$.

Proof: Start with a border Waring rank decomposition

$$p = \lim_{\varepsilon\to 0} \begin{pmatrix} \ell_1(\varepsilon) & \ell_2(\varepsilon) & \cdots & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdots \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) \\ \vdots \\ \ell_r(\varepsilon) \end{pmatrix}$$

$$\overset{\text{every monomial is a power}}{=} \lim_{\varepsilon\to 0} \begin{pmatrix} \ell_1(\varepsilon) & \ell_2(\varepsilon) & \cdots & \ell_r(\varepsilon) \end{pmatrix} \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \begin{pmatrix} \ell_1(\varepsilon) \\ \vdots \\ \ell_r(\varepsilon) \end{pmatrix}$$

$$\overset{\text{Nisan}}{=} \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

$$\overset{p \text{ is a symm. tensor}}{=} \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

□

Remark: Computation via the trace is not closed

Recall:

For an order d tensor t the **tensor complexity** $\mathrm{w}_\otimes(t)$ is defined as the smallest r such that $\exists$ linear forms with $t =$

$$\begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

Nisan: $\forall t$ we have $\underline{\mathrm{w}_\otimes}(t) = \mathrm{w}_\otimes(t)$.

Definition

For an order d tensor t the **trace complexity** $\mathrm{trw}_\otimes(t)$ is defined as the smallest r such that $\exists$ linear forms with $t =$

$$\mathrm{trace}\left(\begin{pmatrix} \ell_{1,1,1} & \cdots & \ell_{1,r,1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,1} & \cdots & \ell_{r,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d} & \cdots & \ell_{1,r,d} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d} & \cdots & \ell_{r,r,d} \end{pmatrix}\right)$$

Theorem [Bläser, I Mahajan, Pandey, Saurabh 2020], confirming a conjecture by Forbes

$\exists t$ such that $\underline{\mathrm{trw}_\otimes}(t) < \mathrm{trw}_\otimes(t)$.

The highest known gap is just 1.

The gap is small, which can be seen by computing summands independently: $\forall t: \; \mathrm{w}_\otimes(t) \overset{!}{\leq} \left(\mathrm{trw}_\otimes(t)\right)^2 \leq \left(\mathrm{w}_\otimes(t)\right)^2$.

A sequence $(c_n)_{n\in\mathbb{N}}$ of nonnegative integers is called **polynomially bounded** if there exist a univariate polynomial t such that $\forall n \in \mathbb{N}: \ c_n \le t(n)$.

For a sequence of multivariate polynomials $(p_n)_{n\in\mathbb{N}}$ we have several sequences of nonnegative integers:

- $\deg(\,(p_n)_{n\in\mathbb{N}}\,) := \big(\deg(p_1), \deg(p_2), \ldots\big)$ degree of the polynomials
- $\mathrm{WR}(\,(p_n)_{n\in\mathbb{N}}\,) := \big(\mathrm{WR}(p_1), \mathrm{WR}(p_2), \ldots\big)$ Waring rank of the polynomials
- etc

Definition (p-family)

A sequence of polynomials (p) is called a **p-family** if $\deg(p)$ is polynomially bounded.

Example: The permanent polynomial $\mathrm{per}_n := \sum_{\pi\in\mathfrak{S}_n} \prod_{i=1}^n x_{i,\pi(i)}$, $\deg(\mathrm{per}_n) = n$.

Remark 1: In the original def, the number of variables must be polynomially bounded, but for all questions of this talk the number of **essential** variables will be polynomially bounded, which is a cleaner, basis independent notion.

Remark 2: We only consider homogeneous polynomials in this talk.

Definitions (**VW**, **VC**, **VBP**) (the V stands for L. Valiant)

VW is the set of p-families with polynomially bounded WR.
VC is the set of p-families with polynomially bounded CR.
VBP is the set of p-families with polynomially bounded w.

- The Shioda polynomial: $s_n := x_1^{n-1}x_2 + x_2^{n-1}x_3 + \cdots + x_n^{n-1}x_1 + x_0^n$ $(s) \in$ **VW**
- The determinant: $\mathrm{det}_n := \sum_{\pi\in\mathfrak{S}_n} \mathrm{sgn}(\pi) \prod_{i=1}^n x_{i,\pi(i)}$ Nontrivial: (det) $\in$ **VBP**
- "(per) $\notin$ **VBP**" is called **Valiant's conjecture**.
  This is the flagship conjecture in algebraic complexity theory, basically an "algebraic P vs NP"
  (per) can be replaced by any so-called VNP-complete sequence, for example the order 4 hyperpfaffian.

Border complexity classes

Recall

VW is the set of p-families with polynomially bounded WR.

VC is the set of p-families with polynomially bounded CR.

VBP is the set of p-families with polynomially bounded w.

Definitions ($\overline{\textbf{VW}}$, $\overline{\textbf{VC}}$, $\overline{\textbf{VBP}}$)

$\overline{\textbf{VW}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{WR}}$.

$\overline{\textbf{VC}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{CR}}$.

$\overline{\textbf{VBP}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{w}}$.

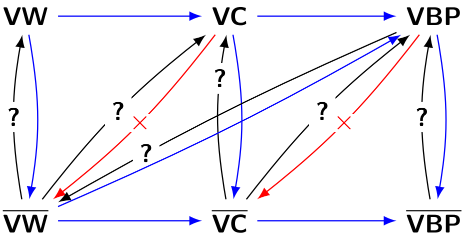
One can define a topology on the set of all p-families such that $\overline{\textbf{VW}}$ is the closure of **VW** etc [I, Sanyal 2021].

Valiant's conjecture 1979: $(\mathrm{per}) \notin \textbf{VBP}$

Mulmuley-Sohoni's conjecture 2001: $(\mathrm{per}) \notin \overline{\textbf{VBP}}$

Is $\textbf{VBP} = \overline{\textbf{VBP}}$? This would imply that the questions at the heart of algebraic complexity theory are questions about algebraic geometry!

Notably we have no candidates for elements in $\overline{\textbf{VW}} \setminus \textbf{VW}$ or $\overline{\textbf{VC}} \setminus \textbf{VC}$ or $\overline{\textbf{VBP}} \setminus \textbf{VBP}$.
Work in this direction by [Grochow Mulmuley Qiao 2016]



$A \to B$ means $A \subseteq B$.

$A \not\to B$ means $A \not\subseteq B$.

- **VC** $\not\subseteq$ $\overline{\textbf{VW}}$ via $\underline{\mathrm{WR}}(x_1 \cdots x_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^{n/2}$ [Landsberg Teitler 2009]
- **VBP** $\not\subseteq$ $\overline{\textbf{VC}}$ is a recent breakthrough via "lopsided flattenings": [Limaye, Srinivasan, Tavenas 2021]

The open questions about this partially ordered set of 6 elements:

1. **VW** $\stackrel{?}{=}$ $\overline{\textbf{VW}}$
2. **VC** $\stackrel{?}{=}$ $\overline{\textbf{VC}}$
3. **VBP** $\stackrel{?}{=}$ $\overline{\textbf{VBP}}$
4. $\overline{\textbf{VW}}$ $\stackrel{?}{\subset}$ **VC**
5. $\overline{\textbf{VC}}$ $\stackrel{?}{\subset}$ **VBP**
6. $\overline{\textbf{VW}}$ $\stackrel{?}{=}$ **VBP**

Interpolation

Theorem (algebraic definition of $\underline{\mathrm{WR}}$)

Let $V = \mathbb{C}[x_1, \ldots, x_n]_d$.
For $p \in V$ we have $\underline{\mathrm{WR}}(p) \leq r$ if and only if there exists a curve

$$p^{(\varepsilon)} \;:=\; p + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots + \varepsilon^e p_e$$

in V with $\mathrm{WR}(p^{(\varepsilon)}) \leq r$ for all ε.

From $\underline{\mathrm{WR}}$ to WR:

View $g(\varepsilon) = p^{(\varepsilon)}$ as a univariate polynomial in ε of degree $\leq e$.
Lagrange interpolation on $e+1$ points:

$$g(\varepsilon) = \sum_{j=0}^{e} g(\alpha_j) \Bigg(\prod_{\substack{0 \leq m \leq e: \\ m \neq j}} \frac{\varepsilon - \alpha_m}{\alpha_j - \alpha_m} \Bigg). \qquad\qquad p \;=\; g(0) = \underbrace{\sum_{j=0}^{e} g(\alpha_j) \Bigg(\prod_{\substack{0 \leq m \leq e: \\ m \neq j}} \frac{-\alpha_m}{\alpha_j - \alpha_m} \Bigg)}_{\text{Waring rank } \leq r}$$

Hence we get a Waring rank $r \cdot (e+1)$ expression.

Therefore: If there exists a bivariate polynomial $q(r, d)$ such that we can always find the $\underline{\mathrm{WR}}$ decompositions so that $e \leq q(\underline{\mathrm{WR}}(p), d)$, then $\mathbf{VW} = \overline{\mathbf{VW}}$.

The best we have is an exponential upper bound on e due to [Bürgisser 2004, 2020] (Lysikov), based on [Lehmkuhl-Lickteig 1989], which goes via proving bounds on the degree of the curve $p^{(e)}$.

The Landsberg-Teitler tables

[Landsberg Teitler 2010] *On the Ranks and Border Ranks of Symmetric Tensors*, Section 10, contains tables:

Border Waring rank 1 approximating curves: $e = 0$

$$x^d$$

Border Waring rank 2 approximating curves: $e = d$

$$x^d, y^d$$
$$x^d, (x + \varepsilon y)^d$$

Border Waring rank 3 approximating curves: $e = 2d$

$$x^d, y^d, z^d$$
$$x^d, (x + \varepsilon y)^d, z^d$$
$$x^d, (x + \varepsilon y)^d, (x + 2\varepsilon y + \varepsilon^{\mathbf{2}} z)^d$$

Border Waring rank 4 approximating curves: $e = 3d$

$$x^d, y^d, z^d, w^d$$
$$x^d, (x + \varepsilon y)^d, z^d, w^d$$
$$x^d, (x + \varepsilon y)^d, z^d, (z + \varepsilon w)^d$$
$$x^d, (x + \varepsilon y)^d, (x + \varepsilon y + \varepsilon^2 z)^d, (x + \varepsilon^2 z)^d$$
$$x^d, (x + \varepsilon y)^d, (x + \varepsilon y + \varepsilon^2 z)^d, w^d$$
$$x^d, (x + \varepsilon y)^d, (x + \varepsilon y + \varepsilon^2 z)^d, (x + \varepsilon y + \varepsilon^2 z + \varepsilon^{\mathbf{3}} w)^d$$

If this pattern continues (i.e., the highest exponent of ε equals the border rank minus 1), then $\overline{\mathbf{VW}} = \mathbf{VW}$.
Note that this is **not** asking for a complete classification of all curves, just about a bound on the exponent.

It is sufficient to look at high degrees

In the following proposition, the function $\gamma : \mathbb{N} \to \mathbb{N}$ can be arbitrary (in particular fast-growing).

Proposition

If there exists a function $\gamma : \mathbb{N} \to \mathbb{N}$ and a univariate polynomial q such that for all $r \in \mathbb{N}$ and all polynomials p of degree $> \gamma(r)$ with $\underline{\mathrm{WR}}(p) = r$ we have $\mathrm{WR}(p) \leq q(r)$, then $\overline{\mathbf{VW}} \subseteq \mathbf{VW}$ (and hence $\overline{\mathbf{VW}} = \mathbf{VW}$).

Proof:
Recall $\underline{\mathrm{WR}}(x^a y^b) = \min\{a, b\} + 1$.

$\underline{\mathrm{WR}}(x_0^{\delta-d} p) \leq (d+1)r$.

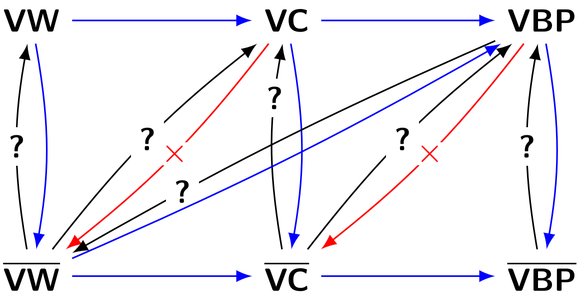
Choose $\delta > \gamma(r)$: $\mathrm{WR}(x_0^{\delta-d} p) \leq q((d+1)r)$.

Set $x_0 = 1$ to get an affine sum-of-dth-powers decomposition of p with $\leq q((d+1)r)$ summands.

Since p is homogeneous, we can take the homogeneous parts of the linear polynomials: $\mathrm{WR}(p) \leq q((d+1)r)$.

For sequences of polynomials with polynomially bounded degree and $\underline{\mathrm{WR}}$, it follows that WR is also polynomially bounded.

□



Thank you for your attention!